

semester

4

Mathematics - II: Teaching Mathematics

COURSE GUIDE

Associate Degree in Education /
B.Ed. (Hons) Elementary

2012



Higher Education Commission

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Higher Education Commission

Foreword

Teacher education in Pakistan is leaping into the future. This updated Scheme of Studies is the latest milestone in a journey that began in earnest in 2006 with the development of a National Curriculum, which was later augmented by the 2009 National Professional Standards for Teachers in Pakistan and the 2010 Curriculum of Education Scheme of Studies. With these foundations in place, the Higher Education Commission (HEC) and the USAID Teacher Education Project engaged faculty across the nation to develop detailed syllabi and course guides for the four-year B.Ed. (Hons) Elementary and the two-year Associate Degree in Education (ADE).

The syllabi and course guides have been reviewed by the National Curriculum Review Committee (NCRC) and the syllabi are approved as the updated Scheme of Studies for the ADE and B.Ed. (Hons) Elementary programmes.

As an educator, I am especially inspired by the creativity and engagement of this updated Scheme of Studies. It offers the potential for a seismic change in how we educate our teachers and ultimately our country's youngsters. Colleges and universities that use programmes like these provide their students with the universally valuable tools of critical thinking, hands-on learning, and collaborative study.

I am grateful to all who have contributed to this exciting process, in particular the faculty and staff from universities, colleges, and provincial institutions who gave freely of their time and expertise for the purpose of preparing teachers with the knowledge, skills, and dispositions required for nurturing students in elementary grades. Their contributions to improving the quality of basic education in Pakistan are incalculable. I would also like to thank the distinguished NCRC members, who helped further enrich the curricula by their recommendations. The generous support received from the United States Agency for International Development (USAID) enabled HEC to draw on technical assistance and subject matter expertise of the scholars at Education Development Center, Inc., and Teachers College, Columbia University. Together, this partnership has produced a vitally important resource for Pakistan.

PROF. DR SOHAIL NAQVI
Executive Director
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How this course guide was developed

As part of nationwide reforms to improve the quality of teacher education, the Higher Education Commission (HEC) with technical assistance from the USAID Teacher Education Project engaged faculty across the nation to develop detailed syllabi and course guides for the four-year B.Ed. (Hons) Elementary and two-year Associate Degree in Education (ADE).

The process of designing the syllabi and course guides began with a curriculum design workshop (one workshop for each subject) with faculty from universities and colleges and officials from provincial teacher education apex institutions. With guidance from national and international subject experts, they reviewed the HEC Scheme of Studies, organized course content across the semester, developed detailed unit descriptions, and prepared the course syllabi. Although the course syllabi are designed primarily for Student Teachers, they are a useful resource for teacher educators too.

In addition, participants in the workshops developed elements of a course guide. The course guide is designed for faculty teaching the B.Ed. (Hons) Elementary and the ADE. It provides suggestions for how to teach the content of each course and identifies potential resource materials. In designing both the syllabi and the course guides, faculty and subject experts were guided by the National Professional Standards for Teachers in Pakistan 2009 and the National Curriculum 2006. The subject experts for each course completed the initial drafts of syllabi and course guides. Faculty and Student Teachers started using drafts of syllabi and course guides and they provided their feedback and suggestions for improvement. Final drafts were reviewed and approved by the National Curriculum Review Committee (NCRC).

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Date of NCRC review: 3 March 2012

NCRC Reviewers: Dr Imran Yousuf, Arid Agriculture University Rawalpindi; Dr Tayyab, Foundation University, Islamabad.

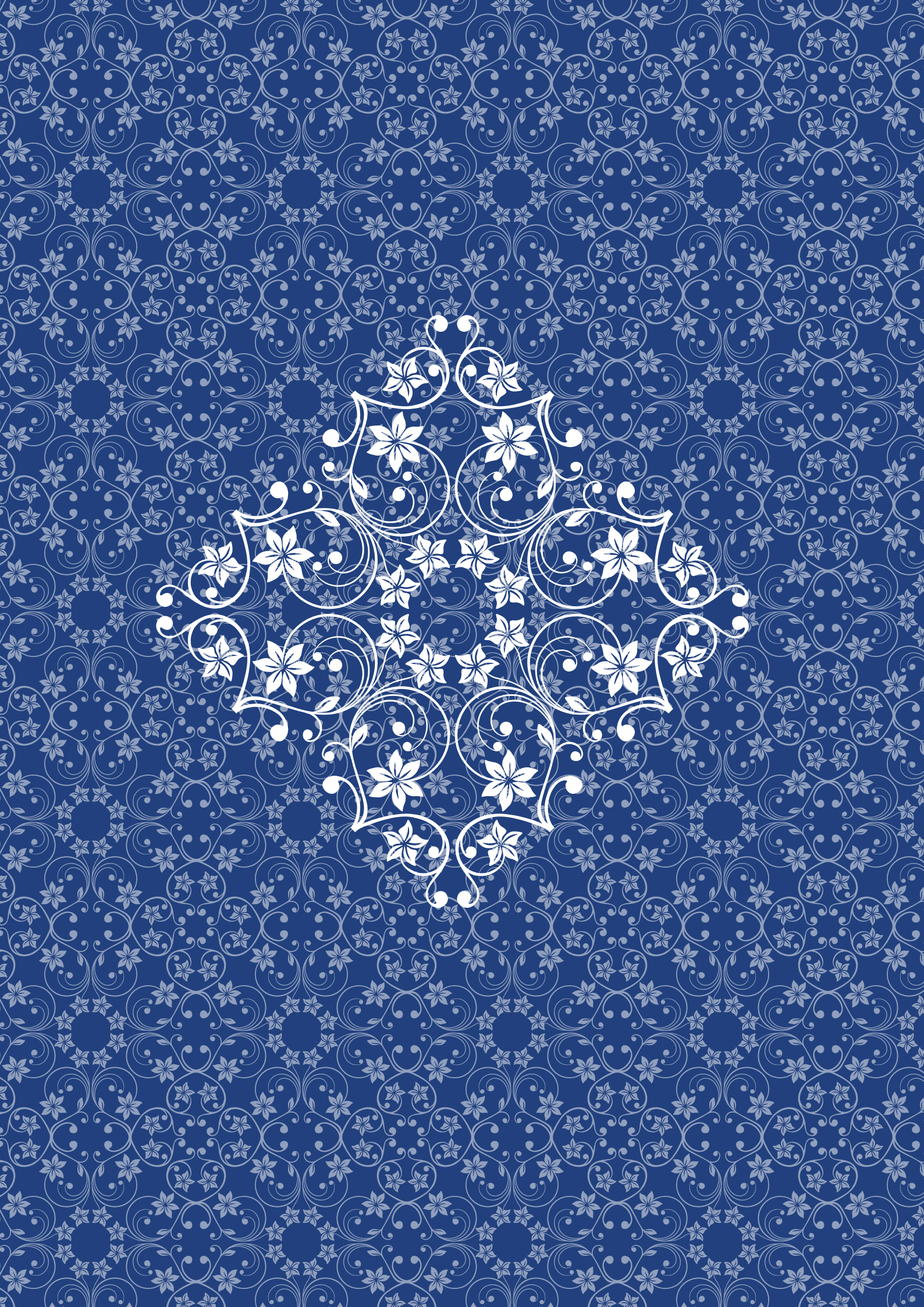


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Syllabus



MATHEMATICS II:
TEACHING MATHEMATICS

MATHEMATICS II

Subject

Teaching mathematics

Credit value

3 credit hours

Prerequisites

Successful completion of the Mathematics I: General Mathematics course in Semester 2

Course description

This course will equip Student Teachers with the knowledge and skills to teach maths in elementary grades. They will become familiar with the mathematics content in Pakistan's National Curriculum and expected student learning outcomes. Student Teachers will learn to use a variety of instructional methods that promote active learning of maths, including making and using teaching and learning materials. They will plan mathematics lessons and activities, and engage in practice teaching of maths.

Course outcomes

Student Teachers will be able to:

- demonstrate a deep understanding of key mathematical concepts in Pakistan's National Curriculum for mathematics in elementary grades
- identify and assess areas of youngsters' understanding and misconceptions to inform their teaching practices
- begin using the pedagogical skills and competencies required to teach mathematics in elementary grades
- describe the nature, history, and development of mathematics education in Pakistan and internationally.

Course structure

Each three-session week will focus on three aspects of maths education: mathematical content, learning the maths content, and teaching the maths content. These will be combined to form an integrated instructional model that addresses the above learning outcomes.

Mathematics content

The first session of each week will begin with Student Teachers working on at least one mathematical problem. They will engage in solving and discussing the problem and sharing approaches and solutions. The maths content will be developed so that Student Teachers will engage in mathematics in depth to help them connect concepts within and across the four units of the National Curriculum: numbers and operations, algebra and algebraic thinking, geometry and geometric measurement, and information handling.

Learning and pedagogy

The week will continue with an emphasis on children's learning and teachers' instructional practices. Class participants will continue to do mathematics to experience approaches to teaching and learning that they can use when they teach. They will recognize that there are often multiple ways of approaching a problem (and in some instances more than one correct answer). The Instructor will present questions that stimulate curiosity and encourage Student Teachers to investigate further: by themselves, with their classmates, or in local schools.

The course will examine how children learn and develop mathematical understanding and skills and how the way children think should influence the teaching of mathematics in the elementary grades.

Assignments

Student Teachers are expected to continue learning about mathematics and the teaching of mathematics after class. There will be assignments to stretch their content knowledge so that they learn more about teaching maths. Assignments will take many forms, including independently solving maths problems as well as school-based tasks.

In summary, the Teaching Mathematics course is a comprehensive effort so that Student Teachers will:

- 1) build and deepen their maths content knowledge
- 2) study ways in which young students learn mathematics
- 3) learn about and use high-quality instructional practice.

Semester outline

Week	Mathematics content	Learning the maths content	Teacher decision-making: Teaching the maths content
1	Prime and composite numbers Factors and multiples	Anticipated student misconceptions	Setting goals for: <ul style="list-style-type: none"> • The programme • Teaching • Learning
2	Division of whole numbers	Emergent mathematical thinking	Lesson design model <ul style="list-style-type: none"> • Launch • Explore • Summarize
3	Prime factorisation Greatest common factor	The value of student errors	Using questioning techniques, wait time, probes, and prompts to foster student thinking
4	Operations with fractions I Least common multiple	Learning mathematics with manipulatives and visual aids	Using problems to develop algorithms
5	Operations with fractions II	Mathematical problem-solving strategies	Physical set-up of a student-centred maths classroom
6	Fractions, decimals, per cents	Mathematical discourse: Learning by talking	Designing and managing cooperative group work
7	Pie charts	Connections between units of the National Curriculum	Timing of lessons, pacing of units
8	Geometric ratios	Maintaining cognitive demand of mathematical tasks	Selecting worthwhile mathematical tasks
9	Proportional thinking	The balance between concepts and skills, the role of drill and practice	Bloom's Taxonomy of Learning applied to mathematics
10	Linear functions and simultaneous linear equations	Multiple representations for a single mathematical concept	Comparing models of teaching I <ul style="list-style-type: none"> • Deductive-analytic • Inductive-synthetic
11	Symmetry	Mathematical learning styles and modalities	Comparing models of teaching II <ul style="list-style-type: none"> • Heuristic • Interactive • Hands-on

Week	Mathematics content	Learning the maths content	Teacher decision-making: Teaching the maths content
12	Volume and surface area	Learning mathematics by writing	Comparing models of teaching III <ul style="list-style-type: none"> • Problem-based learning • Project-based learning
13	Measurement and precision	Precision in mathematical vocabulary and syntax	Differentiated instruction
14	Estimation and large numbers	Learning mathematics with available technology	Differentiated instruction: Assessments
15 and 16	Introduction and/or review of seminal thinkers in mathematics and mathematics education		

Suggested resources

These following websites provide many ideas for teaching and learning mathematics and information about maths education and the mathematical topics addressed during the course.

National Council of Teachers of Mathematics, 'Illuminations'

➤ <http://illuminations.nctm.org>

New Zealand Ministry of Education, 'New Zealand Maths', Curriculum.

➤ <http://nzmaths.co.nz>

University of Cambridge, 'NRICH: Enriching Mathematics'.

➤ <http://nrich.maths.org>

Suzanne Donovan, John Bransford, eds. *How Students Learn: History, Mathematics, and Science in the Classroom* (Washington DC: National Academies Press, 2005).

➤ www.nap.edu/catalog.php?record_id=10126#toc

Nancy Protheroe, 'What Does Good Mathematics Instruction Look Like?'

➤ <http://www.naesp.org/resources/2/Principal/2007/S-Op51.pdf>

Tom Basserear, *Mathematics for Elementary School Teachers* (Belmont, CA: Brooks/Cole, 2012).

Derek Haylock, *Mathematics Explained for Primary Teachers*, 4th ed. (Thousand Oaks, CA: SAGE Publications, 2010).

J. A. Van de Walle, K. Karp, and J. Bay-Williams, *Elementary and Middle School Mathematics: Teaching Developmentally* (Boston: Pearson Education, 2013).

Planning guide

About the planning guide

This Course Guide includes a syllabus and planning guide. The syllabus is primarily for Student Teachers. The planning guide is intended for faculty teaching the course.

The planning guide provides suggestions, resources, and links to resources for teaching the course by week, except for Weeks 15 and 16.

For each week, the planning guide provides faculty notes that introduce and discuss the content for teaching and learning. After the faculty notes, three session plans are offered.

Each session plan is organized into five parts:

- Essential content and activities
- Essential mathematical understandings
- Essential activities
- Discussions about learning and teaching
- Assignments

Faculty notes and session plans contain links to useful resources for faculty and Student Teachers. Some of the session resources are also reproduced in the planning guide.

The planning guide is a guide only. Faculty should use their professional judgement to decide if and how to use the planning guide, and to modify and adapt the content for their context.

The writers of the course aimed to respect copyright and have sought permission to use copyrighted material when necessary. Please contact EDC in case of questions or concerns about any of the materials used, at www.edc.org.

WEEK 1

Week 1: Faculty notes

Maths content

Prime and composite numbers, factors and multiples

Student learning issues

Anticipating student misconceptions

Teaching the maths content

Goal setting: For programme, for teaching, for learning

Look through the following websites:

- Factorize (rectangle applet):
 - <http://tinyurl.com/Rectangle-Factors>
- Factor game (an inductive approach) (Math Solutions):
 - <http://tinyurl.com/Factor-Game-1>
- Factor game (a more deductive approach) (NCTM Illuminations):
 - <http://tinyurl.com/Factor-Game-2>

Download and print out the following handouts (one copy per Student Teacher):

- ‘Mathematical Reflections: My Mathematics’:
 - <http://tinyurl.com/MyMaths-Survey>
- ‘How Many Rectangles?’ recording sheet:
 - <http://tinyurl.com/HowManyRectangles>
- 30 x 40 grid paper (special size to allow for drawing the rectangles 1 x 1 to 1 x 30):
 - <http://tinyurl.com/30x40grid>
- ‘Sieve of Eratosthenes’ worksheet:
 - <http://tinyurl.com/Sieve-Worksheets>
- ‘Prove or Disprove’ prime numbers worksheet:
 - <http://tinyurl.com/Sieve-Worksheets>
- ‘Set of Four Factor Game Gameboards’ (page 1):
 - <http://preview.tinyurl.com/Gameboards>
- ‘Analysing Moves for the 1 through 30 Factor Game’ recording sheet:
 - <http://tinyurl.com/Analyze1-30Factor>
- ‘The Factor Game 49 Board’:
 - <http://tinyurl.com/Factor-Game-Q>

Materials to copy (for Instructor’s use):

- Transparency of the ‘Factor Game’:
 - <http://tinyurl.com/TheFactorGame>

Materials to bring to class: Coloured pencils or markers

This week begins with some very basic concepts about numbers:

- Prime and composite numbers
- Factors and multiples

During this first week of the course, Student Teachers will engage in three dynamic activities designed to help them think about the basics of number theory. Because they may think of number theory as abstract and dry, by the end of this week they will have learned ways of teaching multiplication, division, factors, and multiples by sketching, colouring, and playing a game.

Although prime and composite numbers and factors and multiples are basic concepts, children tend to confuse these terms. Thus, before teaching these topics, it is important for Student Teachers to understand how children think about these ideas and anticipate common misunderstandings when these ideas are introduced.

While the examples of factors and multiples in this week's lesson relate to whole numbers, they also prepare children for the work they eventually will do with operations on fractions. Furthermore, even though factors and multiples may be considered part of the 'Numbers and Operations' unit, they have a strong connection to algebra in later grades. Thus, while children are developing their number sense with whole-number factors, they are also developing the concepts and skills they need when they begin factoring algebraic expressions.

The activities for the week include two hands-on activities that involve drawing rectangles of various side lengths (and then creating a chart of the results to differentiate between prime and composite numbers) and playing a game involving factors and multiples.

In each case, the goal is not simply to engage children in fun activities but to have Student Teachers realize that research has shown that if the above topics are taught by rote definition and a few examples at the board, children usually will not understand the concepts and forget in the future what they have committed to memory today. They then become confused between the factors and multiples, forget which word is which, or consider the ideas hard. This is why we are using research-based activities to inform Student Teachers of typical misconceptions before they occur so that they can prevent them.

This emphasis on children's misunderstandings is the learning concept of the week. It is likely that there are topics that Student Teachers were taught as children they might not fully understand. Thus, it is important that they not only learn the mathematics they will need to teach but also uncover their own misconceptions and confusions that they have held since childhood. Because they might feel embarrassed to show this in front of their Instructors and peers, it is important for your college or university classroom to be considered a safe place to ask questions, and where tentative thinking is honoured and mistakes are explored. By doing this you, as a faculty member, will be modelling the type of teaching you want your Student Teachers to use when they enter their own classrooms.

For this week the third component, instructional practices, is goal setting. As might be expected, many novice teachers (who can become overwhelmed by the demands of their new profession) often think first and foremost about their teaching goal: getting through the lesson. However, there are two other goals they need to address: programme goals and student learning goals.

Programme goals relate to the National Curriculum as a whole, including the unit under study at a particular grade level and how that unit (for example, ‘Numbers and Operations’) at a particular grade level relates to content in the grades before and after. A new teacher needs to know what his or her colleagues taught—and what students should have learned. Similarly, he or she is responsible for preparing students for next year’s teacher. Teachers often assume they need to reteach rather than review or that they need to teach a topic to a high level of mastery, when their job is simply to introduce a topic so that next year’s teacher can continue to teach the same concept in greater depth.

If education can be considered a service profession, then the curriculum and the teacher’s instructional practices exist to serve the student. Most Student Teachers believe this. But once they become employed, many issues challenge their idealism: lack of time, money for materials, textbooks, conflicts among staff members, building conditions, and poverty.

This is why it is crucial for you to emphasize that student learning goals need to be first and foremost in their minds. Once student learning goals are clear, teachers can ‘work backward’ to find teaching practices that answer the questions about instructional decision-making:

How will I help my students learn this concept?

- What do I know about my students?
- Which pedagogical approach *will I choose* to teach this topic?
- Which activity *will I choose* to help my students better understand a concept?
- Which assessment method *will I choose* so that students can show their level of understanding?

Notice that in each question, teachers are asked to consider options and make a choice based on what they know about their students, the maths concept, pedagogical options, different activities, and assessment options.

Finally, this session begins with a student self-assessment, ‘Mathematical Reflections: My Mathematics’. It is intended to give you as their Instructor information about issues that may surface during this course. We have found it useful for Student Teachers to have an informal discussion on the questions before turning in the form. This allows them to add details to their form as they hear topics raised by their peers before turning it in to you. Although this survey is anonymous, it should provide you with a profile of areas you will want to emphasize during the course.

Week 1: Essential content and activities

Student Teacher survey:

'Mathematical Reflections: My Mathematics'

Maths content

Prime and composite numbers, factors and multiples/products

Student learning issues

Anticipating student misconceptions

Teaching the maths content

Goal setting: For programme, for learning, for teaching

Self-assessment ('My Mathematics')

Have Student Teachers complete the 'Mathematical Reflections: My Mathematics' self-assessment, which is intended to alert you to content issues that may surface during this course.

If possible, have your classroom arranged in groups of four to six so that Student Teachers can have an informal discussion before turning in their forms.

Essential mathematical understandings

Any natural number other than one is either prime or composite.

A prime number is any number whose only factors are one and itself.

Composite numbers are natural numbers composed of more than two factors.

Square numbers have an odd number of factors.

Zero and one are neither prime nor composite. Two is the only even prime number.

Factors and multiples have an inverse relationship with each other.

Factors and multiples (as well as prime and composite numbers) can be modelled by finding the dimensions and area of rectangles.

The Sieve of Eratosthenes is a tool that can be used to find prime numbers.

Essential activities

Rectangle lesson

Without introducing the words *prime* or *composite*, have Student Teachers draw all the rectangles with areas of 1 to 30 on the special-size grid paper that can be downloaded from the link in 'Faculty Notes'.

After they have done this, help Student Teachers identify common misconceptions children (and perhaps the Student Teachers) may have by asking questions such as:

- When you drew a rectangle, did you wonder about its orientation? For example for a rectangle with an area of 10, should you have drawn both 2×5 and 5×2 rectangles?
- If you drew both rectangles, did you wonder if the factors of 10 were 2, 5 as well as 5, 2? Is there a difference?
- What about squares? Are they rectangles?
- How would you define a rectangle? Does a square fit that definition?
- When you looked at the factors for a given rectangle, how could you 'work from the ends' to create factor pairs?
- Think about the number of factor pairs for each rectangle (which includes squares). Was there anything different for squares?
- What about the rectangles that can be only 'one high' or 'one wide'?

At this point, introduce the terms *prime* and *composite* if they have not arisen in the discussion already. Continue asking questions about prime and composite numbers, such as:

- How would you define a prime number? (Do not give a definition at this point. Instead ask the following questions so that Student Teachers can develop a definition.)
- What about one?
- What about zero?
- Are all even numbers composite?
- What about two?
- Are all odd numbers prime?
- Are all prime numbers odd?
- What about integers? Do you think -5 is prime? Is -12 composite?
- Can we come up with a definition of prime and composite numbers?

Sieve of Eratosthenes

To further Student Teachers' understanding of prime and composite numbers, begin by asking the following questions:

- What is a sieve?
- What 'drains through'?
- What is 'left behind'?

Distribute the 'Sieve of Eratosthenes' worksheet and have Student Teachers colour the chart according to the directions. (Note: It is better for Student Teachers to tag the corner of a numbered cell with a colour rather than colour in the entire cell. This allows prime numbers eventually to be coloured in fully, and thus stand out from the composite numbers.)

After Student Teachers have completed this task, ask questions such as:

- How did you use your multiplication and division facts to find primes up to 100?
- If you had a calculator, how could you find a prime up to 400? Up to 900? Why do you think I suggested those numbers?
- What general rule would help you decide if a number is a prime number?

The Factor Game

When preparing to teach the 'Factor Game', read through both the inductive approach (<http://tinyurl.com/Factor-Game-1>) and the deductive approach (<http://tinyurl.com/Factor-Game-2>).

With whichever instructional model you choose, begin by using the transparency of the gameboard (at <http://tinyurl.com/TheFactorGame>) to play the factor game: Instructor versus class.

As the game progresses, note Student Teachers' responses. After finishing the first game, play the game a second time to see if their strategies improve.

Finally, have Student Teachers play several games with a partner (using the handout with the four gameboards). Then have them fill out the 'Analysing Moves' recording sheet (at <http://tinyurl.com/Analyze1-30Factor>).

Have a whole-class discussion using ideas from whichever of the lesson plans (inductive or deductive) that you used.

As an extension, ask the following question related to the factor game (from: <http://tinyurl.com/Factor-Game-Q>).

'Long ago, people observed the Sun rising and setting over and over at about equal intervals. They decided to use the amount of time between two sunrises as the length of a day. They divided the day into 24 hours. Why was 24 a more convenient choice than 23 or 25?'

Discussions about learning and teaching

Children's misconceptions

Much of this week's focus on children's learning issues—anticipating student misconceptions—will emerge while Student Teachers are engaged in the activities and games. Their own questions and confusions as young adults will mirror those of the children they soon will be teaching.

Student Teachers may feel, however, that to prevent misunderstandings they will need to be directive as teachers, defining terms and concepts ahead of time. However, this is precisely what causes much of children's confusion! Although counter-intuitive, it is often wiser for teachers to allow confusion to surface during the playing of a game so that misconceptions can be addressed in a class discussion.

Student Teachers' understanding

At the end of the week, ask Student Teachers to assess what they learned during the week about prime and composite numbers (as well as factors and multiples) that they hadn't learned in the primary grades.

To help Student Teachers organize their thoughts, reflect about their understanding, and to observe how their knowledge has moved forward, create a T-chart on the board.

On one side, write down what Student Teachers say they did not learn about these topics when they were children. On the other side, write what they now understand about factors and multiples.

Then ask how this week's activities may have contributed to changing their understanding of these ideas.

Goal setting

As you move through the week, you may want to address the three types of goals (programme, teaching, and learning) each a day at a time. By this fourth semester, Student Teachers should be familiar with the National Curriculum and recognize how topics included in this course are aligned with the material they will eventually need to teach. The Curriculum reflects Pakistan's national goals of what the country believes a youngster needs to know and be able to do as a Pakistani citizen of the 21st century. As such, the Curriculum must be taken seriously.

Student Teachers may ask, however, how closely the Curriculum as a pacing guide needs to be followed. This, then, becomes an issue of teaching goals: 'How do I get through the material? And how do I get through it efficiently while keeping an eye on the important third goal, student learning?' It is well and good for a teacher to check off that he or she has taught the material. But the key question that should be asked is: 'Has my teaching facilitated short-term student learning in the here and now, as well as deeper conceptual learning that these children will need in future grades?'

WEEK 2

Week 2: Faculty notes

Maths content

Division of whole numbers

Student learning issues

Emergent mathematical thinking

Teaching the maths content

Lesson design: The launch-explore-summarize model

Read the following articles:

- A. Downton, 'Links Between Children's Understanding of Multiplication and Solution Strategies for Division':
➤ <http://tinyurl.com/Linking-Division>
- 'Planning a Math Unit: Launch-Explore-Summarize Teaching Model':
➤ <http://tinyurl.com/LaunchExploreSummarize>

Look through the following websites:

- Division as repeated subtraction (compared to multiplication as repeated addition):
➤ <http://tinyurl.com/Div-Subtr>
- Divisibility rules lesson:
➤ <http://tinyurl.com/Div-Rules>

Download and print out the following handouts (one copy per Student Teacher):

- 'Classification of Word Problems' chart, partitive and measurement models of division:
➤ <http://tinyurl.com/ClassifyM-DProb>
- 'Multiplication and Division Word Problems':
➤ <http://tinyurl.com/MultDivWordProb>
- 'Interpreting Remainders':
➤ <http://tinyurl.com/InterpretRemainders>
- 'Divisibility Test' and 'Divisibility Rules':
➤ <http://tinyurl.com/DivisibilityTest> and <http://tinyurl.com/RulesDivisibility>

Materials to copy (for Instructor's use):

- Answer key for 'Multiplication and Division Word Problems':
 - <http://tinyurl.com/MultDivWordProbKey>

Materials to bring to class:

- Small objects to be used as manipulatives to model division
- Grid paper
- Index cards or sticky notes for the divisibility rules activity

Rather than beginning this week's 'Faculty Notes' with maths content, begin by considering the student learning issue for the week: children's emergent mathematical thinking. The article 'Links Between Children's Understanding of Multiplication and Solution Strategies for Division' addresses the intellectual development children need to go through as they learn about division. Because the article discusses fine points of research, it may be helpful to note certain sections that can help Student Teachers understand:

- how even very young children can begin to understand division
- that there is a developmental sequence beginning with young children understanding 'fair shares' (if there are eight crayons and four children, each child receives two) to solve division problems
- how older children use facts and procedures to solve division problems.

As you read the article, the material on the following pages can help you build ideas about both content (division of whole numbers) and learning (children's developmental thinking about a mathematical topic) into your classes this week:

- 'Links Between Children's Understanding of Multiplication and Solution Strategies for Division'
 - p. 174: Division word problems
 - p. 175: Sequence of division strategies from basic to more sophisticated
 - pp. 176–177: Interview examples

Note that research has discovered that the inverse relationship of multiplication and division is a) more difficult to understand than the inverse relationship between addition and subtraction and b) that this understanding develops relatively slowly. This is why teachers need to take an emergent thinking approach to maths topics, not expecting children to understand them abstractly as soon as they are introduced. Teachers should be accustomed to providing students with many opportunities to learn the same concepts. Research literature indicates that learners need between 12 and 20 repetitions to internalize a concept.

This week's maths content, division of whole numbers, extends the work done last week with factors and multiples. When drawing a rectangle with whole-number sides on grid paper, the two sides are factors and the area is the product. But when looking at the same rectangle while thinking about division, the area is the dividend, one of the sides is the divisor, and the other side is the quotient.

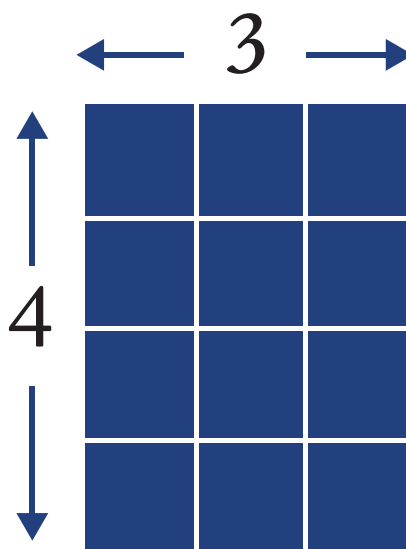


Figure: Rectangle divided into four rows and three columns, equalling 12 parts

This inverse relationship (presented visually above) can also be seen in 'fact families' for division and multiplication:

$$4 \times 3 = 12 \quad 3 \times 4 = 12 \quad \text{and} \quad 12 \div 4 = 3 \quad 12 \div 3 = 4$$

Most children begin to think of multiplication as repeated addition. But only rarely is division introduced as repeated subtraction. This model of division as repeated subtraction is presented to Student Teachers and explored this week.

A conceptual aspect of division is the difference between 'partitive division' and 'measurement division' examples. Student Teachers will work with both types of division word problems in a format similar to the Cognitively Guided Instruction (CGI) models of addition and subtraction to which they were introduced in the Mathematics I (General Mathematics) course. It is important to ask Student Teachers after they work with these problems which ones they think children would find easier or more difficult, and why. This 'why' is essential to help teachers think about children's developmental thinking.

As Student Teachers think about whole-number division, you might ask:

- What is the difference between $35 \div 5$ and $5 \div 35$?
- What word problems could be used to model each?
- Could 5 chairs (a discrete situation) be shared by 35 people?
- Could a piece of yarn or string 5 metres long (a continuous situation) be cut into 35 segments? If so, how long would each segment be?

In whole-number division, sometimes the result is another whole number. However, the result may not be a whole number. There may be a remainder. How to interpret and represent the remainder is important for both Student Teachers and children to understand. When giving children a non-contextual division problem such as $15 \div 4$, a teacher may ask them to address the remainder of problem as a) a whole number ($R = 3$), b) a fraction, or c) a decimal. However, when solving word problems and in real-life situations, children need to consider the context. The remainder can be addressed not only as a whole number, fraction, or decimal; it may also be rounded up or rounded down depending on circumstances.

Finally, divisibility rules for single-digit divisors are something children usually are taught to memorize. However, in the 'Divisibility Rules' activity (in Essential Activities), there is a way for children (and Student Teachers) to use what they know about patterns to develop divisibility rules for 2, 3, 4, 5, 6, 9, and 10.

This week's teaching the maths content, lesson design—the launch-explore-summarize model—will serve as a planning guide for you as you think about structuring your own sessions this week. By using it in your own work, you will be modelling how Student Teachers can use it when they plan lessons in their own classrooms. In fact, one of the assignments this week is for Student Teachers to read 'Planning a Math Unit: Launch-Explore-Summarize Teaching Model' and plan a lesson on division for children in the third or fourth grade.

One major advantage of a launch-explore-summarize lesson is that it allows for several tasks to be completed by different groups during class. In this way, all the tasks are performed by the class as a whole, but not every child has to do every task. It is during the summary portion of the lesson that the results of all the tasks are shared and discussed so that children have a fuller picture of a concept.

Week 2: Essential content and activities

Maths content

Division of whole numbers

Student learning issues

Emergent mathematical thinking

Teaching the maths content

Lesson design: The launch-explore-summarize model

Essential mathematical understandings

Multiplication and division are inverse operations.

Division can be thought of as repeated subtraction.

There are two models of division: partitive and measurement.

If there is a remainder in a division problem, there are several ways to express the remainder, depending on the problem's context.

Divisibility patterns can help children quickly assess division by single-digit numbers.

Essential activities

Cognitively Guided Instruction

Ask Student Teachers if they have ever heard the words *partitive* and *measurement* as applied to division. Most likely they have not. Rather than your attempting to define these two ways of thinking about division, distribute the CGI 'Classification of Word Problems' chart and have Student Teachers attempt in a small group discussion to explain the differences in their own words.

Interpreting remainders

Distribute the 'Interpreting Remainders' handout and have Student Teachers discuss in their small groups the most appropriate way to interpret the remainder given the context of the word problem. Should they express it as:

- a whole-number remainder?
- a fraction?
- a decimal?
- or should they round the quotient up or down to a whole number?

To help Student Teachers interpret remainders, ask questions such as:

- An array implies a rectangular arrangement. How can you use arrays to show a remainder? Thirteen divided by 4 can be arranged as a 3 x 4 rectangle or array with one extra square that lies outside the rectangle or array. This 'extra' represents the remainder.
- If division and multiplication are inverses, what multiplication problem relates to $32 \div 5 = 6 \text{ R}2$?
- Why can there not be a remainder of 5 or 6 when the divisor is 5?

Divisibility rules

The goal of these activities is for Student Teachers to develop divisibility rules, not memorize them. Introduce the divisibility rules worksheets ('Divisibility Test' and 'Divisibility Rules') and ask questions such as:

- How do you know if a number is divisible by 2, 5, or 10?
- What about divisibility by other numbers such as 3, 6, and 9?
- How do the factors 2 and 3 influence divisibility by 6?
- Is there a quick way to test if 2394 is divisible by 9?

Discussions about learning and teaching

Children's emergent mathematical thinking

Share with Student Teachers some of the research findings and children's solution strategies from the article 'Links Between Children's Understanding of Multiplication and Solution Strategies for Division'. To help them discuss children's emergent thinking as it relates to division, ask questions such as the following that highlight the development of children's understanding of simple fair shares to more complex solution strategies:

- Why do you think children understand fair shares as an early way of thinking about division?
- How is skip-counting backward on a number line for division similar to skip-counting forward on a number line for multiplication?
- What does a child need to know to use repeated subtraction as a solution strategy for division problems?
- How can children learn to keep track of skip-counting or repeated subtraction to arrive at a correct answer?
- What prior knowledge and conceptual development do children need to benefit from working with divisibility rules?
- Can you give an example of a division word problem that uses the phrase 'how many times more than'? What do you think children understand by this phrase? When do you think children can interpret this meaning of division?
- When is it appropriate to begin giving children division problems involving a remainder?

Lesson design

During the week's second session, after Student Teachers have read 'Planning a Math Unit: Launch-Explore-Summarize Teaching Model', have a whole-class discussion comparing this model of lesson design to the way maths lessons were structured when they were first learning about factors, multiples, prime numbers, and division.

During the last session of the week, have Student Teachers discuss in small groups the division lessons they planned for third- or fourth-grade children using the launch, explore, and summarize design. Ask them to focus on how this lesson plan may have been different from other lesson plans they have written. Which section did they find most challenging to plan? Why?

Assignments

After Session 1

Have Student Teachers read 'Planning a Math Unit: Launch-Explore-Summarize Teaching Model':

➤ <http://tinyurl.com/LaunchExploreSummarize>

After Session 2

Have Student Teachers plan a division lesson for third- or fourth-grade children using the launch-explore-summarize lesson design.

WEEK 3

Week 3: Faculty notes

Maths content

Prime factorization, Greatest Common Factor

Student learning issues

The value of student errors and error analysis

Teaching the maths content

Questioning techniques

Look through the following websites:

- Questioning strategies for maths:
 - <http://tinyurl.com/Math-Q-Tech>
- ‘Error Pattern Analysis’ protocol:
 - <http://tinyurl.com/Analyze-Errors>
- ‘Math Errors: Learn from Them’ (the value of student errors):
 - <http://tinyurl.com/Value-Errors>
- ‘Factor Trees’ (interactive applet to demonstrate finding prime factors, GCF, and LCM):
 - <http://tinyurl.com/PFact-GCF-LCM>
- ‘The Venn Factor’ (GCF and Venn diagrams to be used as the basis for the GCF lesson):
 - <http://tinyurl.com/GCF-Lesson-Plan>

Download and print out the following handouts (one copy per Student Teacher, except for the factor cards):

- ‘Error Pattern Analysis’ protocol:
 - <http://tinyurl.com/Analyze-Errors>
- ‘Error Analysis: Munira’s Work Samples’:
 - <http://tinyurl.com/8-Work-Samples>
- ‘Factor Trees’, prime factorisation worksheets: Please see the course resources.
- Factor cards (one set for half the Student Teachers):
 - <http://tinyurl.com/VennFactorCards>
- Venn factor worksheets:
 - <http://tinyurl.com/VennFactorWorksheet>
- Venn factor challenge:
 - <http://tinyurl.com/VennFactorChallenge>

Materials to bring to class:

- Yarn or string (to make the Venn diagrams)
- Scissors

The maths content this week, prime factorization and the Greatest Common Factor (GCF), builds on Student Teachers' understanding of prime and composite numbers in Week 1 and the concept of division of whole numbers in Week 2. It also looks ahead to the use of these ideas when working not just with whole numbers but also with fractions beginning in Week 4, when the Least Common Multiple (LCM) will be addressed. (Just as children tend to confuse factors and multiples, they also confuse the GCF and LCM.)

The maths content activities for the week begin by using factor trees to model prime factorization. Student Teachers will use both the factor tree model and Venn diagrams to acquire conceptual knowledge of the GCF.

As Student Teachers work with factor trees, they will note that for a number such as 12, some may begin with 3 and 4 and then use 2 and 2 to factor the 4. This will result in the prime factors 3, 2, 2. Other Student Teachers may begin factoring 12 as 2 and 6, and then factor the 6 into 3 and 2 for a prime factorization of 2, 3, 2. Have them note that both sets of factors contain exactly the same prime numbers, but that it is convention to order them from least to greatest: 2, 2, 3. Student Teachers might notice that there are 'two 2s' in the list and ask if the two 2s could be represented by 2^2 . The answer is yes, giving the prime factorization of 12 as $2^2 \cdot 3$.

This is an appropriate place to introduce the 'fundamental theorem of arithmetic'. Ask if any other number could be represented by a prime factorization of $2^2 \cdot 3$. Conversely, can 12 be represented by any other set of prime factors? Ask them to consider the prime factorization of other numbers. Does each number have its own unique set of prime factors? Is this true for all positive integers (other than 1)?

Having used prime factorization to decompose composite numbers, the results can be used to find the GCF for two or more positive integers numbers.

The student learning issue emphasized this week is the value of error analysis when teachers look at students' work (and listen to students while they work). Teachers need to know the mathematics not only to discover errors but also to analyse why they may be occurring. As mentioned in Week 1, errors might be misconceptions commonly held by children of a certain age. On the other hand, an individual student may display a pattern of errors that is a clue to his or her thinking.

There are two types of error analysis. One occurs in real time while students are in class. The other occurs when a teacher is reviewing student work samples to note a) what a student already knows and b) what he or she does not yet understand, and why. It is only then that the teacher can create targeted instruction to help the student better understand the concept.

Often teachers discover errors by walking around the room to monitor students as they work. Sometimes when detecting errors a teacher might work with students, one at a time, doing individualized tutoring. Although this is a tempting strategy, teachers can usually work one-to-one with only a few children during a class period. It is more time-effective for a teacher to note common student errors while circulating around the room and save them for a whole-class discussion so that all children can get the benefit of review and/or reteaching.

At other times, a student's verbal response in a class discussion may involve an error. This is where the teacher's approach to student mistakes becomes important. Rather than dismissing the answer as wrong and moving on to another student who may have the correct answer, a teacher can keep a light tone and comment, 'I'm glad you said that. In fact, let's talk about it some more, because I'm sure several others of you thought that way, too'.

The 'Error Pattern Analysis' protocol handout can be a starting point for looking at students' work. When analysing a page of student work samples, a teacher scores the paper overall, then looks at the first mistake, attempting to determine the cause of the error or the algorithm that the youngster is using to solve the problem. Then the teacher looks at the next incorrect answer to check if the child is making the same type of mistake. While doing an error analysis it is also important to notice what a student does know in order to use those strengths when devising a plan for remediation.

To practice error analysis, Student Teachers will do an error analysis using the handout 'Error Analysis: Munira's Work Samples', which is related to last week's maths topic, the division of whole numbers.

The teaching the maths content emphasizes this week is on the use of questions to probe Student Teachers' thinking (which can lead to identifying errors in a student's thinking). Deliberately use a variety of questioning techniques during the first two class sessions, without alerting Student Teachers that you are actively doing this.

Be aware that some of the Student Teachers may find this approach somewhat frustrating, especially if they expect you to respond to their questions with direct answers or tell them if their answer was either right or wrong. This feeling of frustration will be true for the children that these Student Teachers will eventually teach, too. In fact, children may feel that unless the teacher gives them a direct answer (or tutors them individually, as described above), the teacher is not helping them. Classroom teachers need to guide children to understand that help means that a teacher will ask questions that prompt and probe a child's thinking and then allow the child time to think.

During the last session this week, ask Student Teachers if they have noticed anything about your questioning style during the past week.

At this point it can be helpful to distinguish between two types of questions: those that prompt, and others that probe. A prompt can be used to offer a hint to a youngster who is stuck without telling him or her the next step in the problem. A prompt may refer to a prior correct example ('What did you do here that got you to the correct answer?') or ask the student to recall a concept studied earlier that can be a clue to solving the problem ('What are some factors of 12? How can they help you figure out 12's prime factors?').

A probing question, on the other hand, asks a child to explain his or her thinking. In this way a teacher can extend the child's thinking by a follow-up question, or use the child's answer to probe more deeply into a misunderstanding.

At the conclusion of this discussion on questions, have Student Teachers work in small groups to create new questions that could be added to the list. Have groups share their ideas for the benefit of the whole class.

Week 3: Essential content and activities

Maths content

Prime factorisation, greatest common factor

Student learning issues

The value of student errors and error analysis

Teaching the maths content

Questioning techniques

Essential mathematical understandings

Whole numbers greater than one can be factored into other whole numbers. Some of these factors may be prime numbers; others may be composite numbers.

Every whole number greater than one can be written as the product of a unique set of prime numbers. This is called the fundamental theorem of arithmetic.

A number's prime factors can be displayed either by using the multiplication sign or by using exponents to indicate the number of times a prime factor is used.

The Greatest Common Factor is the greatest number that is the common divisor of two or more whole numbers.

Essential activities

Prime factorisation and factor trees

Begin by creating a factor tree for 48 on the board, and then have Student Teachers work with the two factor tree worksheets. Ask them to compare their work in small groups. Have a whole-class discussion about prime factorization, asking questions such as:

- Which two numbers might we begin with when creating a factor tree for 48?
- Do those numbers need to be prime numbers?
- How do you find the prime factorization when you reach the end of a factor tree?
- When you found the prime factorization of 30 beginning with two different sets of factors, what was alike about the prime factors? What was different?
- Would the prime factorization of 36 be $2^2 \cdot 9$? Why or why not?
- Can 79 be written as the product of prime factors? Why or why not?
- What difficulties might children have if they create factor trees on blank paper rather than on a worksheet such as the one you just used?
- Why might the use of the multiplication symbol in the following diagram be helpful to children who are being introduced to prime factorization? Does organizing the prime factors in one line seem clear? Or confusing?

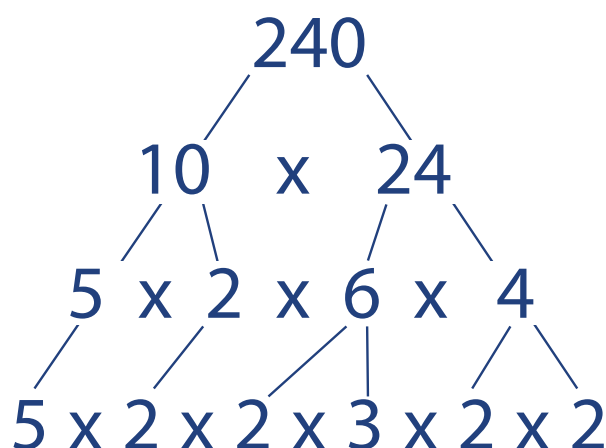


Figure: Factor tree

Greatest Common Factor (GCF)

Begin a class discussion by asking questions such as those below, and then have Student Teachers work in small groups to explore the GCF by using both factor trees and Venn diagrams.

- What does *common* mean? How do you think a child might answer this question?
- How is *a* common factor different from *the* greatest common factor?
- Working backward, the GCF for 24 and 36 is 12. How did I find that?

Factor tree method

Have Student Teachers make factor trees for the numbers 8 and 24. (Note that the prime factors of 8 are 2, 2, 2, while the prime factors of 24 are 3, 2, 2, 2.) Then ask questions such as:

- Which prime factors appear in both these lists? (2, 2, 2) How might you use them to find the GCF?
- Is it important to use all three of the 2s in this example? Why or why not?
- Is it correct to say that all three 2s are prime factors shared between the numbers 8 and 24?
- How is multiplication used to find the GCF once the shared factors are known?

Venn diagram method

At the end of Session 1, send home the Venn factor cards with half the Student Teachers. Have these Student Teachers cut out the cards and bring them to the next class.

Base your lesson on the suggestions found at this 'Illuminations' webpage: <http://tinyurl.com/GCF-Lesson-Plan>. This lesson plan includes:

- Directions for introducing the Venn diagram method
- An explanation of how students will use the small cards they cut out (by placing them into loops of string or yarn that model Venn diagrams)
- Suggestions for numbers to use
- A set of 'challenge questions'
- Follow-up questions

End with a whole-class discussion of the ‘challenge questions’ and by asking questions such as:

- How might you use prime factors to find the GCF of 24 and 36?
- What is the prime factorization of 12? Where do the prime factors for 12 appear in the list of prime factors for both 24 and 36?
- Can you find the GCF for more than two numbers, for example 6, 12, and 48? Why or why not?
- How are prime factors related to the GCF?

Discussions about learning and teaching

The value of student errors and error analysis

Distribute the handout ‘Error Analysis: Munira’s Work Samples’ with eight work samples that relate to operations with whole numbers. Without mentioning any formal protocol, have the Student Teachers follow the directions on the handout as they work in small groups. In each group, have one Student Teacher act as Munira, who will try to explain her thinking. Can group members follow her line of thinking?

After completing the task in small groups, have Student Teachers report on why they think Munira made these mistakes. Then ask questions such as:

- Does Munira know her number facts?
- What does Munira know about place value?
- What sort of aids (drawings, manipulatives, etc.) might help Munira understand division?
- How would they help Munira *without* giving her a procedure?
- How might they use *correct* examples to help her see her mistakes?

Questioning strategies

Ask Student Teachers if they noticed anything about your questioning style during the past week. What questions do they recall your asking? Identify the five types of questions from the ‘Questioning Strategies’ website.

Distinguish between two types of questions: ones that prompt, and others that probe.

At the conclusion of this discussion on questions, have Student Teachers work in small groups to create new questions related to this weeks maths content by using the five categories of questions you identified.

Assignments

After Session 1

Distribute the Venn factor cards (two pages) to half the Student Teachers. Have them cut out the cards and bring them to the next class session.

Have Student Teachers use this interactive applet to review prime factors and how they relate to the GCF and the LCM, with Venn diagrams:

- <http://tinyurl.com/PFact-GCF-LCM>

WEEK 4

Week 4: Faculty notes

Maths content

Operations with fractions I, least common multiple

Student learning issues

Learning mathematics with manipulatives and visual aids

Teaching the maths content

Using problems to develop algorithms

Look through the following websites:

- The pizza puzzle:
 - <http://tinyurl.com/PizzaPuzzle>
- M. Burns, 'Seven Musts for Using Manipulatives':
 - <http://tinyurl.com/Manip-7Musts>
- D. Clements, 'Concrete Manipulatives, Concrete Ideas':
 - <http://tinyurl.com/Manip-Concrete>
- Zeno's paradox (infinite series):
 - <http://tinyurl.com/Zeno-Paradox>
- How to find the LCM:
 - <http://tinyurl.com/LCMreference>

Download and print out the following handouts (one copy per Student Teacher):

- The pizza puzzle (copy the first page only):
 - <http://tinyurl.com/PizzaPuzzle>
- 'Adding Mixed Numbers':
 - <http://tinyurl.com/AddMixedNumbers>
- Fraction strips:
 - <http://tinyurl.com/FractionStrips>
- Pattern blocks template:
 - <http://tinyurl.com/Pattern-Blx>

Materials to bring to class:

- Centimetre grid paper:
 - <http://tinyurl.com/Grid-Paper-Cm>
- Sheets of blank paper (e.g. 12" x 18")
- Graph paper

This week's mathematics content includes addition of fractions along with the Least Common Multiple (LCM; often called the 'Lowest Common Denominator' when dealing with fractions).

Students need to understand the importance of the LCM when adding and subtracting fractions with unlike denominators (such as $3/4 + 5/8$). In primary school mathematics, the LCM may have been called the Least (or Lowest) Common Denominator. In this case of $3/4$ and $5/8$, the LCM is 8. This allows us to rename $3/4$ as the equivalent fraction $6/8$ and add it to $5/8$, with the resulting sum of $11/8$.

The same would hold true for subtraction: the renamed $3/4$ (now $6/8$) $- 5/8 = 1/8$.

Of course, any common multiple could be used as a denominator when adding these two fractions. For example, both could be represented with a denominator of 16 ($12/16 + 10/16$). In this case, the answer ($22/16$) would be renamed in lowest terms as $11/8$.

However, before addressing the LCM, you may want to introduce the topic of adding and subtracting fractions by asking Student Teachers which type of 'whole-number thinking' children may apply to adding fractions such as adding $3/4 + 5/8$. Typically, children will add the numerators and then add the denominators, and assume that the answer is $8/12$ (or perhaps rename it as $2/3$). Although we want children to make generalizations in mathematics, this type of overgeneralization is both common and incorrect. This is because fractions, as a new type of number (rational numbers), do not necessarily follow the same rules that children have internalized about operations with whole numbers.

You may want Student Teachers to think about adding and subtracting fractions by having them engage in activities for adding and subtracting fractions before addressing the LCM. See below for three activities that will challenge Student Teachers to think about this. The first is the pizza puzzle, which has students add fractional parts of a rectangular pizza intuitively, without finding an LCM. (If a more appropriate context is needed, replace the pizza with a rectangular cake.)

Another activity involves adding mixed numbers. Student Teachers are asked to explain their strategy and then to create an explanation as to why answers that appear different are actually equivalent. This should lead to a discussion of equivalent fractions, improper fractions, expressing a fraction in lowest terms, and so forth.

During the previous week, Student Teachers worked with the Greatest Common Factor (GCF) using both factor trees and Venn diagrams. When addressing the LCM this week they will be reminded of another strategy they learned in the Mathematics I (General Mathematics) course: creating lists.

To use the list method to find the GCF between two numbers (for example 16 and 24), we would list all the factors for each number (for 16: 1, 2, 4, 8, 16; for 24: 1, 2, 3, 4, 6, 8, 12, 24), compare the lists, and then look for the greatest factor that appears in both lists. In this case, it would be 8.

To find the LCM for the same numbers (16 and 24) using lists, we would list the multiples for each number (16, 32, 48, 64), (24, 48, 72, 96). In this case the LCM would be 48, the lowest number appearing on both lists.

Another way to find the LCM is to break each number into its prime factors. For 30 and 45, the prime factors would be for 30: 2, 3, and 5; and for 45: 3, 3, and 5. Multiply each factor the greatest number of times it appears when comparing the two lists. In this case there is one 2, two 3s, and one 5. Multiplying $2 \times 3 \times 3 \times 5$, gives the LCM of 90.

The learning topic for the week is the use of manipulatives and other learning aids. Help Student Teachers realize:

- Manipulative materials do not need to be commercial products.
- Manipulative materials can be as simple as a collection of dried beans, water bottle caps, slips of paper, or loops of yarn. All of these free and inexpensive materials can be used to help children understand many mathematical concepts.
- Learning aids can be simple visual handouts downloaded from the Internet for children's use. For this lesson on fractions, the following two PDFs would be helpful visual aids:
 - The fraction strips handout can help both Student Teachers and children visualize equivalent fractions that relate to a common denominator.
 - The pattern block handout can be used by children to colour and cut out models for the fractions $1/1$, $1/2$, $1/3$, and $1/6$ as well as multiples of these unit fractions.

Manipulatives are not going to replace a teacher's own knowledge of mathematics for teaching, however. Create a discussion about the use of hands-on and visual materials with Student Teachers. Ask questions such as:

- What is the purpose of using manipulatives in the classroom?
- Is it to solve a given problem? Or to develop an intellectual model that can be used as a reference in the future? Or both?
- What can you use as manipulatives if you cannot afford to buy commercial products?
- How might you incorporate web-based virtual manipulatives into your future teaching?

This week's teaching focus is using problems to generate algorithms. Usually, the reverse is true: children are given algorithms to practice, and once they are proficient using procedures, they are given problems to solve. These problems, however, are most likely short word problems requiring only the application of a procedure. In essence, they are not 'rich problems' (or puzzles) that require creative thinking. On the other hand, the pizza problem is a rich problem. It presented a situation that Student Teachers probably did not have an algorithm or procedure to solve. By working through the problem, however, they may discover an algorithm that could be used to add fractions. This model of problem-based learning will be explored later in the course, but introducing it now will give Student Teachers a different perspective on doing procedural problems after learning an algorithm versus solving problems (puzzles) that lead to the development of an algorithm or to deeper conceptual understanding.

Week 4: Essential content and activities

Maths content

Operations with fractions I, least common multiple

Student learning issues

Learning mathematics with manipulatives and visual aids

Teaching the maths content

Using problems to develop algorithms

Essential mathematical understandings

Addition and subtraction of fractions do not follow the same rules as the addition and subtraction of whole numbers.

Equivalent fractions can be used when adding and subtracting fractions.

Just as prime factorisation could be used to find the Greatest Common Factor, the same technique can be used to find the Least Common Multiple.

The Least Common Multiple, sometimes called the Least Common Denominator, can be used when adding and subtracting fractions.

Essential activities

Pizza puzzle

As Student Teachers do this problem (from the pizza puzzle handout), have half the groups use grid paper while the other half uses a large sheet of paper that they will fold and label with fractional amounts.

Have Student Teachers do the graphing activity so that they can discuss non-linear graphs.

Ask questions such as the following:

- Based on this 'eaten $1/2$ each day' timetable, could the whole pizza ever be eaten? Why or why not?
- How small could the next piece be?
- Is there a difference between a purely mathematical answer to this problem versus an answer based on cutting the $1/256$ portion in half (Zeno's paradox)?
- How did you add together the fractions of the amount eaten?
- How was the total of the amount eaten related to the amount left? To the whole?
- How do the graph paper and folded paper models compare?
- What does the graph suggest? Is it linear? If not, how would you describe it?

Adding mixed numbers

Have Student Teachers analyse how each group described in the 'Adding Mixed Numbers' handout may have found an answer.

- Which algorithms might each of the different groups have developed?
- As Student Teachers listen to these explanations from their peers, could they follow the line of thinking?
- If not, how might the presenting group edit their explanations to be more clear and precise?
- What issues about equivalence arose?
- How can you prove equivalence?

Continue this focus on adding fractions with unlike denominators by having Student Teachers use pattern blocks that they coloured and cut out for homework. Ask them to consider the yellow hexagon as the whole and to add fractions with the unlike denominators of two, three, and six. For example, what do they discover when adding $\frac{1}{2}$ (modelled by the red trapezium) and $\frac{1}{3}$ (the blue rhombus)? Why does the answer need to be expressed in sixths by the green triangles? This activity paves the way for working with the Least Common Multiple.

Least Common Multiple

Have Student Teachers use the list method to find the first ten multiples of 6 and 4. Then ask questions such as the following:

- What are the common multiples between the two lists?
- What is the Least Common Multiple?
- How could you use this idea of the LCM to add the fractions $\frac{3}{4}$ and $\frac{5}{6}$?
- How does the LCM relate to the addition and subtraction of fractions?
- How could you find the LCM by using factor trees?
- How is the LCM different from the GCF?
- Why do you think children confuse the two?
- Is there a Greatest Common Multiple? A Least Common Factor? Why or why not?

Discussions about learning and teaching

Why manipulatives?

Hold a discussion regarding the use of manipulatives and visual aids. Focus on how a teacher's 'mathematical knowledge for teaching' (a term coined by researcher Deborah Ball) is crucial when a teacher makes a decision as to when and how tactile and visual models can support student learning.

To prompt the discussion, ask questions such as the following:

- How did paper folding and/or drawing on grid paper help your group solve the pizza problem?
- How might the visual of factor 'trees' with their prime factor 'roots' help children understand the difference between the GCF and LCM?
- How did working with the pattern block cut-outs help you understand children's thinking?

Problems and problem-solving

After Student Teachers have worked on the pizza problem and the developing algorithms problem, ask the following questions:

- How do you define a mathematical ‘problem’?
- Do you need to know how to solve a problem before considering it a ‘problem’?
- What is the difference between solving:
 - a traditional textbook’s assignment of 12 computational problems?
 - a real-life mathematical problem?
 - a non-traditional mathematical problem (such as discovering patterns in Pascal’s triangle or finding multiple expressions to describe how to determine the surface area of a cuboid)?
- Which real-world issues involving fractions do you deal with in your own life?
- Which real-world issues involving fractions might be of concern to children?
- How might you, as classroom teachers, build those issues into problems involving fractions before teaching children an algorithm?

Assignments

After Session 1

Have Student Teachers colour the pattern blocks on the pattern blocks handout according to the following scheme, cut them out, and bring them to the next class:

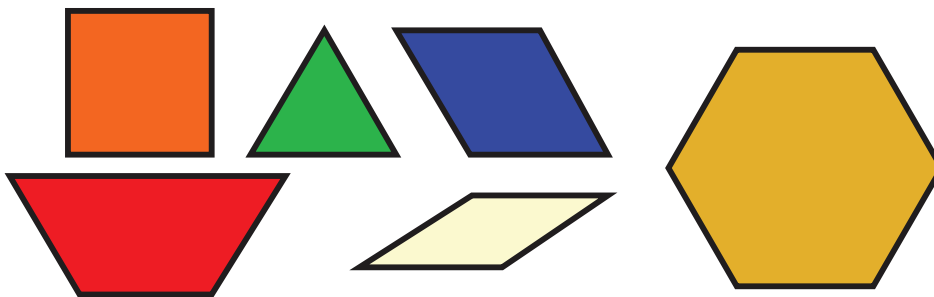


Figure: Example of cut out and coloured pattern blocks.

- Hexagon: Yellow
- Trapezium: Red
- Square: Orange
- Equilateral triangle: Green
- ‘Wide’ rhombus: Blue
- ‘Thin’ rhombus: Tan

Have Student Teachers review this website that addresses the Least Common Multiple:

➤ <http://tinyurl.com/LCMreference>

WEEK 5

Week 5: Faculty notes

Maths content

Operations with fractions II

Student learning issues

Mathematical problem-solving strategies

Teaching the maths content

Setting up a student-centred maths classroom

Read the following articles:

- A. Kohn, 'What to Look for in a Classroom':
➤ <http://tinyurl.com/Look-at-Classroom>
- B. Nelson and A. Sassi, 'What to Look for in Mathematics Classrooms':
➤ <http://tinyurl.com/LookMathClass>
- K. Lim, 'A Collection of Lists of Mathematical Habits of Mind':
➤ <http://tinyurl.com/MHoM-Lists>

Look through the following websites:

- Area model for multiplication of fractions:
➤ <http://tinyurl.com/Mult-Fract-Area-Model>
➤ <http://tinyurl.com/Mult-Fract-Area-2>
- Area model for division of fractions:
➤ <http://tinyurl.com/Div-Fract-Area-Model>
 - Division of fractions lesson plan:
➤ <http://tinyurl.com/Div-Frac-Models>
 - Common denominator model for division of fractions:
➤ <http://tinyurl.com/Div-Fract-Com-Denom>

Download and print out the following handouts (one copy per Student Teacher):

- 'Using Rectangles for Multiplying and Dividing Fractions'. Go to <http://tinyurl.com/Mult-Div-Frac-Rect> then download a document listed under 'Attachments' titled 'using rectangles.pdf'
- 'Multiplying and Dividing Fractions on a Number Line': Go to <http://tinyurl.com/Mult-Div-Frac-Rect> then download the document listed under 'Attachments' titled 'multiplying_dividing.pdf'
- 'Why Do We Invert and Multiply?', division of fractions (first page only):
➤ <http://tinyurl.com/Why-Invert>

- ‘Creating Story Problems for Multiplication and Division of Fractions’:
➤ <http://tinyurl.com/Fract-Story-Problems>
- K. Lim, ‘A Collection of Lists of Mathematical Habits of Mind’:
➤ <http://tinyurl.com/MHoM-Lists>
- A. Kohn, ‘What to Look for in a Classroom’:
➤ <http://tinyurl.com/Look-at-Classroom>

Materials to copy (for Instructor’s use):

- Division of fractions lesson:
➤ <http://tinyurl.com/Div-Frac-Models> (pages 1–3)
- Models for dividing fractions: Ice-pops:
➤ <http://tinyurl.com/Div-Frac-Pops>
- Models for dividing fractions: Chocolate bars:
➤ <http://tinyurl.com/Div-Frac-Choc>

Materials to bring to class:

- Coloured pencils
- Scissors

For additional reference:

- A. Cuoco, P. Goldenberg, and J. Mark, ‘Habits of Mind: An Organizing Principle for Mathematics Curriculum’:
➤ <http://tinyurl.com/MHoM-EDC> (a 23-page article)
- ‘Developing Effective Fractions Instruction for Kindergarten through 8th Grade’: <http://tinyurl.com/Fractions-K-8-Report> (A 90-page report that can be slow to load. Cut the URL and paste into your browser for faster loading.)

This week continues last week’s work (addition and subtraction of fractions), moving to the multiplication and division of fractions.

Children first think that whole-number operations apply to the addition and subtraction of fractions—adding or subtracting across the + or – sign, as in $\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$. (Which of course it does not!)

Now, however, when dealing with the multiplication and division of fractions, new rules apply:

- When multiplying fractions, you do multiply across the multiplication sign (such as $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$).
- Even more confusing to children, $\frac{1}{2} \div \frac{1}{4} = 2$. But to get that correct answer, they have to invert the $\frac{1}{4}$, make it a 4, cancel out 2, and then multiply across the ‘top and bottom’ (the numerators and denominators) to arrive at what the teacher says is the correct answer of ‘2’.

All of this is quite confusing to children!

Therefore, teachers need to make what seem to be mathematical contradictions clear to their students. The activities during this week will help Student Teachers understand the underlying mathematics about multiplication and division of fractions so that they can help students be less confused about these two topics.

The learning concept this week addresses strategies that deepen mathematical understanding. These are not strategic tricks that can be used to solve problems. Rather, they are 'Mathematical Habits of Mind (MHoM)'. The collection of MHoM in the short article by Lim is related to the longer 23-page article listed as an additional reference. Although all MHoM are valuable and your Student Teachers should be aware of them, only four will be emphasized in the maths activities this week:

- Visualizing
- Modelling
- Describing
- Looking for patterns

Of course, MHoM are not specific to the multiplication and division of fractions. Rather, they are ways to develop a 'mathematical mindset' that will serve children in any maths area from grade 1 to university level. But in order to develop their students' MHoM, teachers need to be explicit about them, regardless of the maths topic.

If contemporary maths teaching is directed toward student understanding of essential concepts, then observers (principals, supervisors, mentors, and coaches) need to note what to look for when they visit maths classrooms to make sure that mathematics is being taught effectively.

What does a learner-centred classroom look like (compared to a more traditional classroom)? In 'What to Look for in a Classroom', Kohn contrasts these two types of classrooms by means of a chart that observers can use to gauge the overall climate of a teacher's learner-centred instructional style. The chart gives a description of how a student-centred classroom ideally should be arranged. The way a teacher chooses to use classroom space is important. For example, if a classroom has movable desks, they can be arranged in groups that are in a 'U' formation, allowing the teacher to walk among groups while monitoring their work.

But what is the difference between 'looking at a lesson' and 'looking at a maths lesson'? Factors that contribute to the generic look and feel of a student-centred classroom may be necessary – but not sufficient – for the maths-specific content, pedagogy, student-thinking, and intellectual community that characterize high-quality mathematics lessons. Consider what is similar and different about the Kohn and Nelson and Sassi articles.

Week 5: Essential content and activities

Maths content

Operations with fractions II

Student learning issues

Mathematical problem-solving strategies

Teaching the maths content

Setting up a student-centred maths classroom

Essential mathematical understandings

Multiplication and division are inverse operations. This inverse relationship applies to fractions as well as whole numbers.

The operation of multiplication does not always make an expression's product 'greater than' its multipliers.

The operation of division does not always make an expression's quotient 'less than' its divisor and dividend.

The multiplication and division of fractions can be visualized with area models and the number line.

The operation of division can be thought of as multiplication by a number's reciprocal.

Children need to understand the multiplication and division of fractions conceptually, not just procedurally.

Essential activities

Division of fractions simulation

The division of fractions lesson plan stresses visual models and is a complete lesson plan. Because of its visual and interactive nature, it is an ideal activity to start this week's topic.

Please review it and then bring it to class with you to use as a guide. It begins with a simulation of cutting ice-pops and chocolate bars into fractional parts of the whole. After Student Teachers describe what happened in words, they develop number sentences to reflect the division that just occurred. The lesson continues by having Student Teachers draw pictures to understand the more abstract idea of dividing one fraction by another fraction. In each case Student Teachers are asked to make connections among multiple representations: an action (dividing something), words, pictures, and equations, and a general rule or algorithm for division with fractions.

Multiplying and dividing fractions using a number line

This second activity will use the worksheet 'Multiplying and Dividing Fractions on a Number Line'. There are two reasons for using this particular worksheet. The first is that it uses a visual model (the number line) to help students see the operations of multiplication and division of fractions and how these relate to the same operations with whole numbers. It may be useful to draw a number line on the board and do several simple repeated whole-number multiplication and division problems before introducing the worksheet.

Note that each number line is scaled differently so that it relates to the fractions being used. This means that Student Teachers will need to deal with equivalent fractions. Some may find it helpful to label each number line with its common unit.

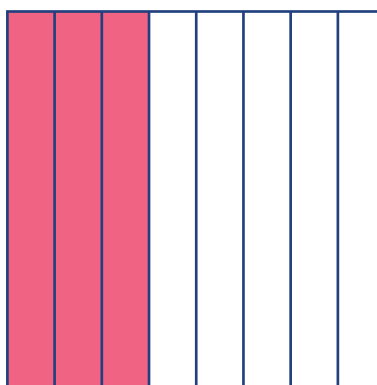
The second reason for using this worksheet is its emphasis on how to phrase division problems and how children need to go beyond just reading the numbers and symbols aloud in left to right order. For example, when given the expression $2\frac{1}{2} \div \frac{1}{2}$, it would normally be read in words as 'two and one-half divided by one-half'. Although this is a literal reading of the numbers and symbols, it does not make sense to children (or many adults!). (This worksheet provides alternate phrasing to help increase children's understanding. Below each division example on this worksheet, there is a different way to describe what is happening in the expression $2\frac{1}{2} \div \frac{1}{2}$: 'How many $\frac{1}{2}$'s are in $2\frac{1}{2}$?)

Another way to phrase the same expression would be ' $2\frac{1}{2}$ divided into halves'. Two points here: first, children can understand 'divided into halves' (that is, multiple portions of $\frac{1}{2}$) more easily than the phrase 'divided by half'. Second, $2\frac{1}{2}$ 'divided into halves' is not the same as $2\frac{1}{2}$ 'divided in $\frac{1}{2}$ ' (which would be $1\frac{1}{4}$).

Multiplying and dividing fractions using rectangles

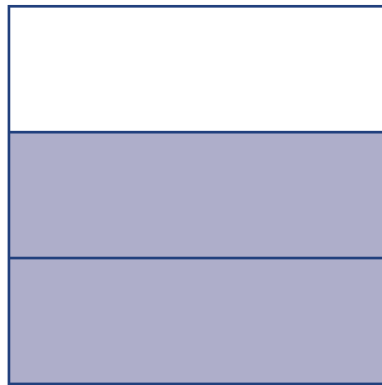
Distribute the 'Using Rectangles for Multiplying and Dividing Fractions' worksheet, on which the area model is used as a way to visualize the multiplication and division of fractions. This worksheet uses the same wording as the previous one, but Student Teachers may need more help visualizing how to colour in the rectangles. One way to do this is to draw models.

Shade one square, partitioned vertically, to represent $\frac{3}{8}$ (shown below in pink):



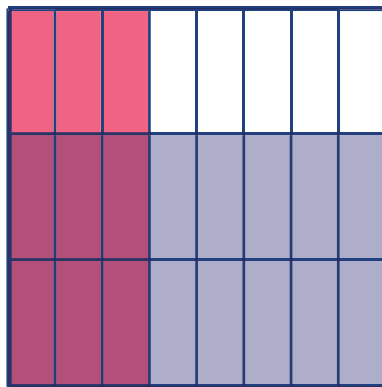
$\frac{3}{8}$

Shade another square, partitioned horizontally, to represent $\frac{2}{3}$ (shown below in blue):



$\frac{2}{3}$

Superimpose the two squares. The product is the area that is double-shaded (shown below in purple):



$$\frac{3}{8} * \frac{2}{3} = \frac{3 \cdot 2}{8 \cdot 3}$$

There are 3×2 , or 6, purple parts out of 8×3 , or 24, parts in all, so the value of the purple area is $\frac{6}{24}$, or its equivalent fraction $\frac{1}{4}$. (Note that the asterisk in the diagram signifies a multiplication sign.)

As Student Teachers work with the rectangles for multiplication, ask if they see a pattern and how that pattern relates to the traditional algorithm they learned in school when they were children.

When they begin working on this area model for division, ask them to ‘think backward’. If multiplication and division are inverse operations (each one ‘undoes’ the other), how can they use what they just learned about fractional multiplication by using rectangles to solve problems for division by fractions?

Have Student Teachers work in small groups to come up with conjectures, and then have a large group discussion about their ideas. Ask how their strategies relate to the traditional algorithm they learned in school.

Finally, ask them to evaluate both models and reflect on their experience with them. Ask questions such as:

- Did these models clarify why the traditional procedures work?
- Was it difficult to use these models because you were already familiar with the traditional algorithm?
- If so, do you think children would find using models difficult?
- How much practice might you need to feel comfortable with these models?
- How much modelling may children need before being introduced to the traditional algorithm?
- How did using these models stretch your own thinking about operations with fractions?

Creating story problems

The handout 'Creating Story Problems for Multiplication and Division of Fractions' asks Student Teachers to work backward from what they know. In this case, they are asked to use their traditional algorithm to solve several multiplication and division problems involving fractions.

In most textbooks, students are asked to practice procedural problems many times and only then are they given word problems to solve. For this activity, however, Student Teachers are asked to create word problems that model the equation they just solved. They will also need to draw visual models that match the equation. This can be a very challenging task. If they find it difficult, it may be wise to give only one multiplication and one division problem to begin with. The important thing is that all six problems are solved with a story and a picture so that they can be analysed and the questions at the end of the worksheet can be used in a whole-class discussion.

Why invert and multiply?

The final activity is reading and interpreting an explanation for 'Why Do We Invert and Multiply?' which ties together all the week's prior work. The goal is for Student Teachers to check, after doing all the week's activities, if they now can make better sense of the traditional algorithm. As they do the reading individually, have them take notes on where the explanation is confusing to them. Have them share their questions in small groups and then open it up for a whole-class discussion.

Discussions about learning and teaching

'Mathematical Habits of Mind'

Distribute the 'Four Mathematical Habits of Mind (MHoM)' handout and have Student Teachers answer the questions individually. After this, have them discuss their responses in small groups.

Then have a class discussion during which each group shares their thoughts. Ask questions such as:

- How did visuals in the activity on rectangles help you understand why you multiply across the numerator and denominator?
- Which models made the traditional algorithms clear for you?
- What vocabulary and syntax helped you clarify how to describe multiplication and division of fractions to students?
- Where did you see patterns about division with fractions?
- Was it challenging for you to work backward? To create a story problem for a given mathematical expression? Why?
- Do you have a better understanding of the reciprocal after doing these activities?

Looking at mathematics classrooms

Distribute the handout 'What to Look for in a Classroom'. In a large group discussion, have Student Teachers compare the two columns in the handout. Ask questions such as:

- Tell me about your own school experience.
- Do you see value in some of the items in the right-hand column? Which ones? Why?
- What about the list in the left-hand column?
 - Why might these be considered student-centred?
 - Which of these do you think are important for teaching maths?
- Do you think your experience learning mathematics would have been different if you had been in a student-centred classroom? If so, why?
- Can you envision setting up a student-centred classroom when you begin teaching?
 - Do you think there will be difficulties?
 - What might they be?
 - How can you anticipate addressing them?
- Which student-centred practices have you seen during this course?

Assignments

After Session 1

Have Student Teachers review the following websites that address the multiplication and division of fractions:

- Area model for multiplication of fractions:
 - <http://tinyurl.com/Mult-Fract-Area-Model>
 - <http://tinyurl.com/Mult-Fract-Area-2>
- Area model for division of fractions:
 - <http://tinyurl.com/Div-Fract-Area-Model>
- Division of fractions lesson plan:
 - <http://tinyurl.com/Div-Frac-Models>
- Common denominator model for division of fractions:
 - <http://tinyurl.com/Div-Fract-Com-Denom>

WEEK 6

Week 6: Faculty notes

Maths content

Fractions, decimals, per cents

Student learning issues

Mathematical discourse: Learning by talking

Teaching the maths content

Designing and managing cooperative group work

Read the following articles:

- L. de Garcia, 'How to Get Students Talking!':
➤ <http://tinyurl.com/Get-Kids-Talking>
- 'Strategies for Promoting and Managing Effective Group Work':
➤ <http://tinyurl.com/Promote-Manage-Group>
- 'Making Small Groups Work':
➤ <http://tinyurl.com/Make-Small-Groups-Work>

Look through the following websites:

- 'Summary of Misconceptions about Decimal Numbers':
➤ <http://tinyurl.com/Decimal-Misconc>
- Fraction and decimal concepts:
➤ <http://tinyurl.com/Frac-Decimal-Concepts>
- Decimal and per cent concepts:
➤ <http://tinyurl.com/Dec-Percent-Concepts>
- Fraction, decimal, and per cent applet:
➤ <http://tinyurl.com/FracDecPerc-Applet>

Download and print out the following handouts (one copy per Student Teacher):

- 'Fraction Strips' (with hundredths):
➤ <http://tinyurl.com/Frac-Dec-Strips>
- 10 x 10 decimal grids:
➤ <http://tinyurl.com/Decimal-Grids>
- 'Misconceptions in Comparing Decimals':
➤ <http://tinyurl.com/Decimal-Misconc-2>
- 'In-Between Game':
➤ <http://tinyurl.com/InBetween-Game>
- 'Connecting Decimals and Fractions':
➤ <http://tinyurl.com/Frac-Dec-Chart>
- 'To Terminate or Not to Terminate':
➤ <http://tinyurl.com/Term-Decimals>

- ‘Strategies for Promoting and Managing Effective Group Work’:
 - <http://tinyurl.com/Promote-Manage-Group>
- L. de Garcia, ‘How to Get Students Talking!’:
 - <http://tinyurl.com/Get-Kids-Talking>

Materials to bring to class:

- Coloured pencils
- Several metre sticks
- Chart paper
- Simple four-function calculators (Student Teachers should bring their own)

This week’s maths topic will help Student Teachers explore rational numbers in three different formats: fractions, decimals, and per cents. The goal is for them to see the relationships (and equivalences) among three different ways to express the same number (for example, $\frac{1}{4}$, 0.25, and 25).

This week’s maths focus is also designed to prepare Student Teachers for next week’s information handling topic: pie charts, in which fractions, decimals, and per cents can be used interchangeably in a circular area model.

This week we continue to use the number line and area (grid) models used in earlier sessions on fractions to show how fractions, decimals, and per cents are related.

These models are important as most adults have been taught only the procedures to convert one format to another (fractions to decimals, decimals to per cents, etc.) without understanding the concept of these equivalent representations.

Although decimals were introduced in the Mathematics I (General Mathematics) course, a word about calculations with decimals is in order. It is interesting that addition and subtraction with fractions is difficult for students because they often use the Lowest Common Denominator as a strategy. Addition and subtraction with decimals, however, is relatively easy.

Addition and subtraction of decimals is done by lining up the decimal points, placing zeros to the right if necessary, and then treating the procedure like whole-number addition or subtraction:

$$\begin{array}{r}
 1.14 \\
 + 5.10 \\
 \hline
 6.24
 \end{array}$$

Multiplication of fractions is a relatively easy procedure (even if children do not understand the underlying concept). Multiplication of decimals, however, is procedurally more difficult, and the answer is often incorrect, especially when children have not estimated the answer by talking through the problem aloud. When children multiply decimals, even those who get the correct digits in the answer often struggle with where to place the decimal point.

Consider multiplying 0.5×0.6 , where the answer is 0.3 . Talking through the problem might sound somewhat like this:

- I know that 0.5 is $\frac{5}{10}$, which equals $\frac{1}{2}$.
- If I take half of 0.6 , which is $\frac{6}{10}$, the answer will be $\frac{3}{10}$ or 0.3 .

This is why it is important for children to understand the connection between fractions and decimals and to be able to move between fractions and decimals easily.

Now consider multiplying 'decimal mixed numbers' such as 1.45×32.6 , which involves inserting a decimal point into the product. Again, it helps to begin with estimation. What will the answer be between? 1.45 is between 1 and 2 . If we were to multiply 32.6×1 , the answer would be 32.6 . If we multiplied 32.6 by 2 , the answer would be about 60 . By estimating, it suggests an answer somewhere between 32.6 and 60 . Therefore the answer would need to be 47.20 . It could not be 472.70 or 4.727 .

When children work with per cents, they are usually taught to 'move the decimal point to the right two places, then add the per cent sign'. Again, this idea of moving decimal points is procedural and does not help children understand the concept of equality when moving between the formats of fractions, decimals, and per cents.

To help children see this equality, creating a reference chart of equivalencies can be useful. One of these is included as an interactive applet that also shows an area model.

The maths activities for the week include modelling fractions and the decimal numbers equivalent to them, as well as playing a game that helps reinforce the place value of decimal fractions that come between any two rational numbers. A second activity explores the idea of repeating and terminating decimals and has Student Teachers chart decimal fraction equivalents. Student Teachers will also discuss an article about common misconceptions that children have about decimals. Finally, they will address the concept of per cents as a ratio based on 100 and how per cents relate to fractions and decimals.

The learning concept this week is mathematical discourse. How can students (either children in primary and secondary classrooms or Student Teachers in this course) 'learn mathematics by talking'?

Questions to consider:

- What is mathematical discourse?
- What does it look like? What are its indicators?
- How can a teacher facilitate it?
- How can I, as a faculty member, plan lessons for my Student Teachers so that mathematical discourse happens in my classroom?

This week's teaching component addresses group work and the decisions teachers need to make when beginning to use cooperative learning in their classrooms. It also relates to this week's learning concept, mathematical discourse, because it is in groups where mathematical discourse happens.

It is important that your Student Teachers understand that simply arranging desks into groups does not mean that group work will occur. Unless the teacher carefully considers what group work entails, children sitting in pairs or quads can simply do individual work on a task that does not require interpersonal mathematical communication.

Teachers need to be aware that a classroom where interactive cooperative learning happens does not happen all at once! Teachers need to prepare their students for the tasks, norms, roles, and interactions that cooperative learning entails. In *Elementary Classroom Management: Lessons from Research and Practice* (2007), the authors Weinstein and Mignano offer the following quote:

Too many teachers think that cooperative learning is putting students into groups and telling them to work together. They select tasks that are inappropriate for the size of the group; they use heterogeneous groups when homogeneous groups would be more suitable (or vice versa); they fail to build in positive interdependence and individual accountability; they fail to appreciate the differences between helping groups and cooperative learning.

Here are some other ideas about cooperative learning to share with Student Teachers:

- The assigned task is important. Many textbook activities are designed for procedural individual work. Thus, a teacher needs to find tasks that require children to work on a conceptual problem with inputs from group members.
- Consider giving only one copy of the task to the group. This is to encourage sharing and support discourse. It will also discourage 'high achievers who may prefer to work alone' (as described by Weinstein and Mignano).
- Deciding on group members is important. Teachers need to know their students in order to decide how to create cooperative groups. It is not wise to have children self-select. Children will either choose their friends or ask to be with a student who seems to 'have all the right answers'.
- Groups require norms of behaviour that need to be posted and then assessed at the end of a lesson. The classroom climate has to be intellectually safe for children to make conjectures (and mistakes) without feeling embarrassed.

- The goal is for each group to contribute to the whole class's understanding. It is not to beat other groups by finding a particular right answer.
- A group needs defined roles for group members. Children need to be trained in knowing what these roles require. Some roles might be procedural:
 - Materials manager
 - Recorder
 - Reporter
- Others roles are more subtle, but more important. Such as:
 - Task introducer (responsible for reading and providing a first interpretation of the task)
 - Group facilitator (responsible for keeping group members focused on the task, not socializing to the point where the task is not completed)
 - Timekeeper (responsible for ensuring that time is being well used)
 - Social monitor (responsible for ensuring respect for all ideas that surface and participation by each group member)
- Teachers need to monitor groups by moving around the classroom and collecting data about student ideas and the group's progress on task. These data can then be used for student presentations during the summary phase of the lesson.

Week 6: Essential content and activities

Maths content

Fractions, decimals, per cents

Student learning issues:

Mathematical discourse: Learning by talking

Teaching the maths content

Designing and managing cooperative group work

Essential mathematical understandings

Fractions are rational numbers expressed in the format a/b .

Some fractions have denominators that are powers of 10, such as:

$3/10$ $83/100$ $283/1000$

These same fractions can be written in place value format:

0.3 0.83 0.283

In this format they are called *decimal fractions* (more commonly known as *decimals*).

Decimal fractions are rational numbers if they are *terminating* (for example 0.3, which equals $3/10$) or *repeating* (for example, 0.3333 recurring, which equals $1/3$).

Non-terminating and non-repeating decimal numbers (for example, pi) are not fractions, and therefore are called *irrational numbers*.

Per cents are fractions based on a denominator of 100. Thus:

- Because of equivalent fractions, $3/10 = 30/100$
- In decimal format, $30/100 = 0.30$
- And in per cent format, $0.30 = 30\%$

Fractions, decimal fractions, and per cents are simply different ways to describe the same number.

Models to connect fractions, decimals, and per cents include:

- The number line (linear model)
- A 10 x 10 grid area model
- The circular (pie chart) model (which will be explored next week)

Just as fractions can indicate mixed numbers, so can decimals and per cents. Thus:

- In decimal format, $2\frac{1}{2} = 2.5$
- In per cent format, $2.5 = 250$ per cent

There can be 'more than 100 per cent' in many real-life situations.

Both children and adults often describe '0.3' in its left-to-right visual sequence: 'zero-point-three'. The correct way to speak about 0.3 is as 'three-tenths'.

Essential activities

Number line and area models for fractions

Some review of decimal fractions will probably be necessary before doing the following activity. In particular, ensure that Student Teachers know that a 'decimal' is not the decimal point. Rather, it is a *number* that can be represented on a number line. This is similar to Student Teachers understanding that a fraction is also a number that can be represented on a number line, not simply the picture of an area model.

To help with this, distribute the handout 'Fractions Strip with Hundredths', in which various fractions, including hundredths, are represented. Use this model to introduce the idea of decimal fraction equivalents. (You can also use metre sticks to model fractions and decimals on a number line.)

Distribute copies of the 10 x 10 decimal grids. Ask Student Teachers what each whole grid could be worth. The word *could* is important, because children, when seeing a 10 x 10 grid, usually assume that the grid is worth 100. It takes a shift in children's thinking to conceive the grid's worth as '1' and each small square inside the grid as worth $1/100$ or 0.01.

Ask Student Teachers how they might help children make this shift in understanding.

At this point, have half the Student Teachers colour in the fractions $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$. Ask the rest of the Student Teachers to colour in 0.5, 0.25, and 0.3333. How do their colourings compare?

Ask Student Teachers to colour in 0.05, then 0.005. What did they have to do to model thousandths?

What would 1.5 look like using decimal grids?

(Have Student Teachers bring these colouring sheets back to class because they will be used later in the week.)

The 'In-Between Game'

Have Student Teachers play the 'In-Between Game' in pairs. They can begin with whole numbers, move on to fractions such as $\frac{1}{2}$ and $\frac{1}{4}$, and finish by playing the game with decimal numbers.

- What happens when the two numbers are 1.0 and 1.5?
- What if the numbers were 1.24 and 1.25?

As Student Teachers work with these types of numbers, have them note when they need to go from tenths to hundredths, and then from hundredths to thousandths.

- Do they understand that there are an infinite number of rational numbers between any two rational numbers?
- Do they understand that each of these 'in between' numbers can be written as either a fraction or a decimal?

Have Student Teachers discuss 'Summary of Misconceptions about Decimal Numbers', the webpage they reviewed for homework. Ask questions such as:

- How did the material relate to some of the activities we did this week?
- Did you hold any of these misconceptions when you were a child? If so, which ones?
- What strategies could you use to help your future students avoid these misconceptions?

Fractions to decimals using calculators

The following activity relates fractions to decimals by working with calculators.

Distribute the handout 'To Terminate or Not to Terminate' and have Student Teachers experiment with their calculators to discover what a fraction in the form a/b 'looks like' when translated into decimal format. Ask questions such as:

- What similarities and differences do you notice?
- What about $1/7$? Does it seem to fit a pattern? How could you test that?
- Do you know how to use long division to find the decimal equivalence for a fraction? (If a fraction is a/b , then it can be thought of as division. When using long division, you continue to divide until the quotient either terminates or shows repetition.)
- What generalisations can you make about fractions and repeating or terminating decimals?
- Can you think of any non-repeating decimal (e.g. pi)? Does it have fractional equivalent? (No, but students often think that pi 'is' $22/7$.)

End by distributing the chart 'Connecting Decimals and Fractions' that asks Student Teachers to find different ways to express the same number. After they have completed the chart, ask questions such as:

- Which of the columns seemed most difficult?
- Were there any rows that you found confusing?
- Did you have to 'work backward' to fill in some of the cells?
- How might you adapt this chart for children? What kind of scaffolding might they need?

Beyond 100 per cent

The mathematics of this section has been kept relatively brief to allow Student Teachers to reflect on the cooperative learning and mathematical discourse that occurred during the week.

Remind Student Teachers that per cents are based on 100, whereas decimals can be based on any power of 10 (10, 100, 1000, etc.). Also remind them that just as with mixed numbers for fractions ($2\frac{1}{2}$) or decimals (2.5), you can have an equivalent per cent (250 per cent) greater than 100 per cent.

Have Student Teachers return to the chart of equivalences ('Connecting Decimals and Fractions') they created in the last lesson, and add a column showing the per cent format for each row.

Have Student Teachers consider how they could use the 10 x 10 decimal grid to model 0.25 per cent and 134 per cent. Ask how each of these could be expressed as a fraction, decimal, or mixed number.

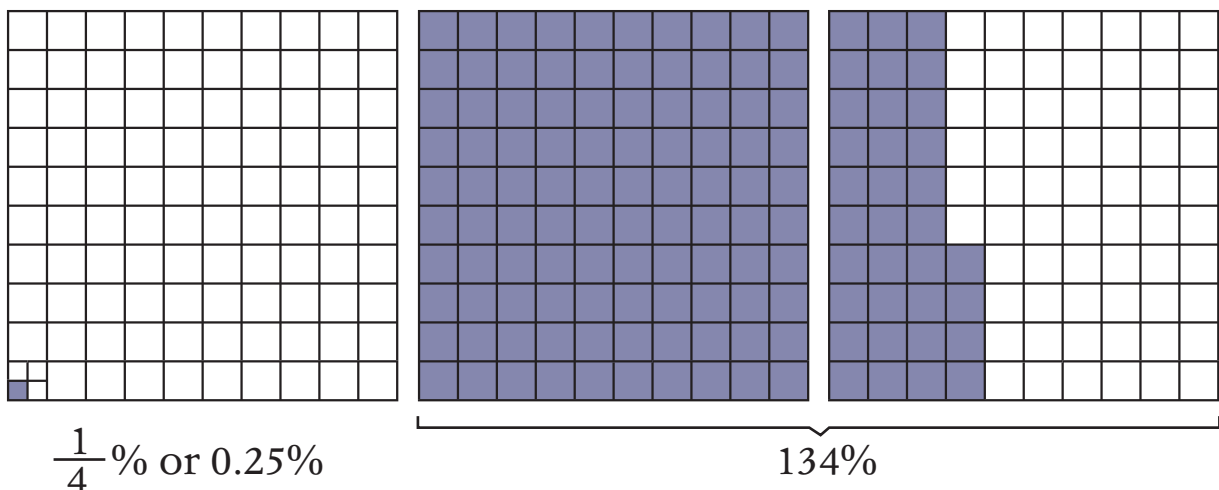


Figure: Ten by ten decimal grid

Ask them to come up with a generalization that would help them convert fractions to decimals to per cents.

(If Student Teachers need to review how to convert fractions to decimals and per cents, there are two websites in the assignments section to help them. The goal for the week, however, has been conceptual understanding, not simply procedures.)

Discussions about learning and teaching

‘Talking’ mathematics

During the second session of the week, facilitate a discussion of ‘How to Get Students Talking!’ that Student Teachers read for homework. Have them share the ideas they highlighted when reading the article. Ask questions such as:

- What types of communication would you hear in a mathematics classroom?
- What does mathematical discourse sound like?
- What are its indicators? Can you give some specific examples?
- What kind of classroom mathematics talk might not be mathematical discourse? Can you give some specific examples?
- How would you define mathematical discourse?
- How can a teacher facilitate mathematical discourse?
- What did you notice in the charts at the end of the article about how mathematical discourse is developed over time, beginning with primary grade children?
- What other norms might be needed when teaching older children?
- Which indicators in the charts could help you foster mathematical discourse in your own classroom when you begin teaching?

Ask other questions related to this week’s maths concepts. In particular, ask how to advance mathematical discourse by having students use precise terminology when talking about rational numbers.

Cooperative group work

Have Student Teachers read the article ‘Strategies for Promoting and Managing Effective Group Work’ in class. (Because of its length, this particular article may be read in a jigsaw fashion). Ask questions such as the following about the tasks they completed in class this week.

- Were some of those tasks more suitable for pair or group work than others?
- Were group norms followed during the tasks?
- Were there additional roles that could have been assigned to help the group stay on task or encourage more active participation?

Assignments

After Session 1

Have Student Teachers read the article ‘How to Get Students Talking!’ to prepare for a class discussion on mathematical discourse. Have them highlight three ideas they thought were important:

- <http://tinyurl.com/Get-Kids-Talking>

Have Student Teachers review 'Summary of Misconceptions about Decimal Numbers' to prepare for a class discussion:

- <http://tinyurl.com/Decimal-Misconc>

If Student Teachers need to review the procedures for converting fractions, decimals, and per cents, have them look at these two webpages:

- <http://tinyurl.com/Convert-Fr-Dec-1>
- <http://tinyurl.com/Convert-Fr-Dec-2>

Have Student Teachers use this interactive applet, which illustrates equivalent fractions, decimals, and per cents:

- <http://tinyurl.com/FracDecPerc-Applet>

WEEK 7

Week 7: Faculty notes

Maths content

Pie charts

Student learning issues

Connections between units of the National Curriculum

Teaching the maths content

Timing of lessons, pacing of units

Read the following article:

- M. Sangster, 'Reflecting on Pace'. Available as MT204 at:
 - <http://tinyurl.com/Reflect-on-Pace>

Look through the following websites:

- 'Understanding Pie Charts' (also includes 'Waffle [100-grid] Charts'):
 - <http://tinyurl.com/PieChart-Design>
- Procedures for creating pie charts:
 - <http://tinyurl.com/PieChart-MiddleGrade>
- Create-a-graph applet:
 - <http://tinyurl.com/Create-a-Grap>
- Create-a-graph tutorial:
 - <http://tinyurl.com/Create-a-Grap-Tutor>
- Everyday Mathematics pacing suggestions:
 - <http://tinyurl.com/EDM-Pacing-Tips>

Download and print out the following handouts (one copy per Student Teacher):

- M. Sangster, 'Reflecting on Pace'. Available as MT204 at:
 - <http://tinyurl.com/Reflect-on-Pace>
- 'Hundredths Disk':
 - <http://tinyurl.com/Hundredths-Disk>
- 'My Home Library':
 - <http://tinyurl.com/MyHomeLibrary>
- 'The 50-Minute Bean Chart' (lesson plan on pacing):
 - <http://tinyurl.com/Pacing-50-Beans>

Materials to copy (for Instructor's use):

- 'Analysing Pie Charts' (four pie charts to analyse) included in course resources

Materials to bring to class:

- Coloured pencils or markers
- Compasses, but not protractors
- Blank computer paper
- Dried beans

NOTE: Student Teachers will need an electronic copy of the National Curriculum to work with the learning and teaching emphasis for this week: connections and pacing.

Pie charts (or circle graphs) are an extension of prior work with fractions, decimals, and per cents. Whereas last week's work focused on the conceptual relationship among these formats, this week's work allows Student Teachers to apply them in a manner that highlights the learning concept of the week: helping children make mathematical and real-life connections among the number, geometry, and information handling topics in the National Curriculum.

Pie charts are used to show percentages of a whole, where each section of the circle should have both a label and a percentage. The total number in the dataset should be indicated in the key. Pie charts are used to display discrete data. They are best used when there are between three and seven categories of data because colour or shading is an important component in making clear the distinctions among and between sections.

Children in the primary grades might be able to interpret simple pie charts by noting which sections appear smaller and which larger. Once children have been introduced to fractions by using a circular area, they can begin to make estimates about the relative fractional sizes of a circle graph's sections.

Although children need to learn from an early age how to read and interpret pie charts, it is a challenge for children in higher elementary grades to construct them. This is where their understanding of the relationship between fractions and percentages becomes important.

In addition, children need to understand that the number of items in the dataset they are considering is 'one whole' and that the circle represents 100 per cent. Because pie charts display discrete (countable) data, children can begin to calculate fractions and percentages of the set's parts once they know the total number of items in the set. This use of part-to-whole is an important concept for children to understand.

Younger children can create pie charts from datasets that require only simple fractions. For example, the dataset in a textbook activity to help children create a pie chart on favourite treats might be $\frac{1}{2}$ of the class favours ice cream, $\frac{1}{4}$ likes biscuits, and another $\frac{1}{4}$ prefers potato crisps.

Constructing pie charts from real-life data that children collect is a much more difficult task. It requires that they not only know how to translate the raw data into fractions and then into per cents, but finally how those per cents relate to the 360-degree measurement of a circle. Even when children can translate fractions to per cents, changing per cents into degrees is a more difficult task. Although there is a procedural algorithm to do this, most children do not understand why it works. What can a teacher do to make creating pie charts less difficult, less algorithmic, and more understandable?

The first, low-tech, method is to have children:

- 1) Trace or draw a circle.
- 2) Lightly, in pencil, divide the circle into fourths (25 per cent, or 90-degrees).
- 3) Again, in pencil and around the edge, divide each of those fourths into four equal sections.
- 4) This will make the distance between each mark approximately 6 per cent, or about 22-degrees.

A second method is to give Student Teachers a handout of a 'Hundredths Disk'. Because all circles are congruent (which children should remember from their study of geometry), they can use the Hundredths Disk to create their pie charts once they have calculated percentages.

A third and high-tech solution is to introduce children with computer access to Create-a-Graph. This app takes a bit of experimentation. Have Student Teachers use the tutorial accompanying the website.

The mathematics activities for the week include two that bridge the gap between last week's focus on fractions, decimals, and per cents and the logistics of translating rational numbers into the 360-degrees of a circle. Both activities are designed to scaffold their last week's learning to prepare them for a complex pie chart task, 'My Home Library', that involves real-life data.

The learning concept for the week is mathematical connections. By making connections, teachers can actually teach several mathematical strands of the National Curriculum at the same time and make connections to children's real-world experiences. This is where applications such as graphs—in this case pie charts—are valuable.

This week the third component, teaching the maths content, is pacing. As mentioned in the first week of the course, when the teaching component was goal-setting, teachers need to pay attention to their teaching goal of getting through their well-planned lesson in the amount of the time they have available. They also need to think of their lesson design. (First of all, do they have a model for lesson design?) One example would be the launch-explore-summarize model discussed earlier in the course:

- 1) Launch (short length of time, whole group)
- 2) Explore (extended length of time, small group)
- 3) Summarize (medium length of time, whole group)

As novice teachers think of these three lesson components, they can benefit from thoughts voiced by experienced teachers:

- ‘Waiting time during the whole-group launch and summary is a good use of time!’
- ‘Giving too many instructions in the launch not only takes more time, but it can overload children with too much information. That’s why I use a fairly short launch.’
- ‘I walk around the room collecting data on all student ideas (so I can build my summary) rather than sitting down with just a few students.’
- ‘I can stop the exploration for a few minutes if several children are confused by the same thing. This allows me to clarify the issue for the whole class.’
- ‘Students need to know how to work in groups so that they stay on task.’
- ‘Materials needed for the task should be quickly available. This is why I always assign a group member to be the materials manager’.

Student Teachers also need to be alert to the overall pacing of the various units and the yearly goals set by the National Curriculum. And as mentioned during the first week, Student Teachers need to become familiar with the whole of the National Curriculum. They need to know the scope of what needs to be taught from years 1 to 8 so they can know the expectations of whatever grade they eventually are hired to teach.

Week 7: Essential content and activities

Maths content

Pie charts

Student learning issues

Connections between units of the National Curriculum

Teaching the maths content

Timing of lessons, pacing of units

Essential mathematical understandings

Pie charts (or circle graphs) are used to display discrete data in the form of fractions or per cents of the whole.

The circle used in the pie chart represents the whole of the dataset: 100 per cent.

Each segment of a pie chart needs a label describing the segment as well as its per cent.

Fractions can be converted to per cents by multiplying the fraction by 100.

Fractions can be converted to degrees by multiplying the fraction by 360.

Essential activities

Introducing pie charts and creating them

Introduce the topic of pie charts, noting that these graphs have a relationship to the discrete data graphs they explored in past courses. However, instead of using actual counts of data (as in tally charts, bar graphs, and line plots), pie charts describe portions of the whole dataset using fractions or per cents.

Note that many pie charts shown in newspapers and on the Internet are poor mathematical examples of circle graphs. For example, they may not indicate the fraction or per cent for each wedge or the sections of the chart may not be labelled. Often there may be too many sections displayed, which can be visually confusing. If possible, show the colour transparency 'Four Pie Charts to Analyse' and ask Student Teachers to assess their usefulness in communication information.

Tell Student Teachers that although they will be working with pie charts this week, they will not be using protractors, a tool they probably were taught to use to create pie charts at school.

More important, do not ask Student Teachers if they remember the procedure used to translate fractions and per cent into degrees. This is something you want them to discover via the work they do during this week.

Instead, ask the following questions:

- How could you create a pie chart if you did not have a protractor?
- Or, how could you create a pie chart if you had a protractor but did not know how to use it?
- Or, if you did not know that there were 360-degrees in circle?
- What do you know about fractions and per cents (and the area model of a circle) that could help you?
- Think about creating a tool that would allow children in higher elementary grades to create pie charts if they could only use their prior knowledge about fractions or per cents. What might that be like?

Building on their responses to the last question, distribute a copy of the Hundredths Disk and ask how it might be useful for children who did not know how to use a protractor or how to convert fractions or per cents into the 360-degrees of a circle.

Have Student Teachers working in small groups use the Hundredths Disk and what they know about fractions and per cents to create one pie chart per group for the following dataset of 100 items: 25, 10, 5, 4, 3, 8, and 45. When they have completed the task, ask questions such as:

- How did using the Hundredths Disk differ from the way you were taught to create pie charts?
- Did it make the task easier? If so, how?
- How did you label the segments of your pie chart? Did you use fractions, per cents, or degrees?
- Why do you think you were assigned to create your pie chart from a dataset of exactly 100 items?
- Were there certain number combinations that made the task easier? What were they? (For example, 25 is one-fourth of the circle, 45 and 5 are one-half of the circle, etc.)
- How can carefully chosen numbers for any topic be helpful in scaffolding tasks for beginning learners?

Contrast the strategies they used to complete this task using the Hundredths Disk versus the way they were probably taught to create pie charts. Ask questions such as:

- What if you had been given a protractor? How might your strategy have been different?
- What if you had access to a calculator? How might your strategy have been different?
- What if you already knew the formula for translating fractions into degrees? How might your strategy have been different?

‘My Home Library’

Distribute the handout ‘My Home Library’. This is a pie chart activity connected to a real-life dataset, meaning it is a more complex problem than the previous one. The goal is to help Student Teachers appreciate that what appear to be relatively simple textbook activities—asking children to create pie charts—can be as complex for children as ‘My Home Library’ may be for adults.

Have Student Teachers work in small groups and answer the questions in the first section, where they analyse the implications of the raw data.

- Were all the numbers needed to answer the questions?
- What calculations did they have to make?
- What inferences did they have to make?

Move to the second set of questions, where Student Teachers are asked to make decisions about the merits of one type of graph versus another.

Did some Student Teachers vote for a bar graph simply because it would be easier to create? Given what they have read and the graphs they have looked at, can they justify why they think a bar graph might communicate information more effectively than a pie chart?

As they go on to create their circle graphs, emphasize that they can use one of the tools they experimented with but that they should be making estimations and sketching the graph, doing the best they can. As you move around the classroom, listen to their strategies for translating the data from the narrative into a pie chart, and check for the following:

- What do Student Teachers need to do to get the data ready?
- If necessary, remind Student Teachers that the 90 and 180 are data, not degrees.
- Once they have the data ready, do they translate the data into the 360-degrees of a circle?
- Or are they using fractional parts to construct their pie chart, and not concerned by the number of degrees?
- How are they sequencing the segments around the circle?
- Are they combining 'friendly' numbers such as 180 and 20 to give half the total number of books (and thereby half the circle)?
- How much estimating is going on? Or are Student Teachers more concerned with using their measuring tool?

In the summary, ask questions such as the following:

- What did you find most difficult about this task?
- How could you have done a rough sketch of the graph if you did not have a measuring tool?
- How would you help children create a pie chart using simpler numbers?
- What might be a useful dataset to introduce children to creating a pie chart?
- How did the numbers used in an earlier session (25, 10, 5, 4, 3, 8, 45) scaffold the 'My Home Library' task for you?
- Did you think of a way to change data into fractions, fractions into decimals, decimals into percentages, and finally percentages into degrees?

Emphasize that this particular task would be very difficult for children in higher elementary grades (even when they are given a textbook dataset whose numbers are easily converted to familiar fractions). It would be far more difficult for children to handle a dataset such as 400 books, 13 of which are poetry.

If no one has raised the issue, mention that there is a formula for translating raw data into degrees when making a circle graph: $\text{angle in degrees} = \text{frequency} / \text{total} \times 360\text{-degrees}$.

Ask questions such as:

- What would be the disadvantage of introducing this formula before children have had time to consolidate their understanding about how fractions, per cents, and angles relate to a circle graph?
- How do you think real data can be made manageable for children?

Discussions about learning and teaching

Before engaging in classroom discussions about connections and pacing, Student Teachers will need to complete several of the homework assignments.

Making connections

Build on this activity after Student Teachers have completed the homework assignment 'Looking for Connections' in the grade 4 maths curriculum.

Ask Student Teachers about the connections in this week's maths content.

- Connections within the four mathematical strands of the National Curriculum (numbers and operations, geometry, algebra, and information handling)
- Connections to other subject areas (literature, science, history, the arts)
- Connections to real-life situations

Reflecting on pacing

Build on the homework assignment to read 'Everyday Mathematics'.

Distribute the article 'Reflecting on Pace'. Have Student Teachers read it in class and discuss in small groups where they agree and/or disagree with the author, and then have a representative from each group share its thoughts with the whole class.

Allocating time during a lesson

The '50-Minute Bean Chart' activity is a way to impress upon Student Teachers how precious their instructional time is. This activity addresses questions such as:

- How can I plan a launch-explore-summarize lesson so there is sufficient time for each segment?
- How long should the launch be?
- How much time does it take to review prior knowledge?
- How do I deal with time spent addressing interruptions in the classroom?
- How much time should be devoted to the exploration phase?
- How do I avoid running out of time so that I can help students summarize the key learning that took place about the lesson's stated objective?

Have Student Teachers recall one of the lessons presented during the course or one they have observed in their classroom visits. Distribute the '50-Minute Bean Chart' and 50 small beans to each Student Teacher.

There is no ideal distribution of time during a 50-minute lesson, but by actually using a manipulative (the beans) students can experiment with different scenarios, helping them ‘see’ how time is spent in a classroom and how, if ‘everything else’ takes up too much of the 50 minutes, instructional time is lost.

Finally, ask them to consider the pacing guidelines in the National Curriculum. Without looking through the National Curriculum, ask, for example, how many days they think should be spent on teaching operations with fractions in the third grade. Should all those days be clustered together? Or might they be spread apart?

Ask Student Teachers to consider how a pacing guide can be helpful, when it can become constricting, and how they will make practical decisions about pacing. Ask questions such as:

- What if students do not have sufficient prior knowledge?
- What if students do not understand a topic even after the lesson is finished?
- How, as teachers, will they determine if they should spend more time on a topic?
- How will they decide when to move on?

Assignments

After Session 1

Have Student Teachers experiment with Create-a-Graph to create pie charts and move between bar graphs and pie charts: <http://tinyurl.com/Create-a-Grap> (It may be helpful to begin with the Create-a-Graph tutorial at: <http://tinyurl.com/Create-a-Grap-Tutor>.)

To prepare for class discussion, have Student Teachers review the fourth grade topics in the National Curriculum, looking for places where they see connections. Mention that they should look both for connections between and among the number and operations, algebra, geometry, and information-handling strands as well as connections to other subject areas. Finally, ask them to think of connections to real-life situations that would be of interest to a fourth grade student.

After Session 2

Have Student Teachers read this list of ‘Everyday Mathematics’ pacing ideas from veteran teachers: <http://tinyurl.com/EDM-Pacing-Tips>. Ask Student Teachers to find one idea they found useful about pacing that they can share in the next class session.

WEEK 8

Week 8: Faculty notes

Maths content

Geometric ratios

Student learning issues

Maintaining cognitive demand of mathematical tasks

Teaching the maths content

Selecting worthwhile mathematical tasks

Read the following article:

- ‘Analyzing Mathematical Instructional Tasks’ by Mary Kay Stein, Margaret Schwan Smith, Marjorie A. Henningsen, and Edward A. Silver:
 - <http://tinyurl.com/MKS-Task-Analysis>

Look through the following websites:

- Dilations PowerPoint presentation:
 - <http://tinyurl.com/PPT-Dilations>
- Finding a good copy:
 - <http://tinyurl.com/A-good-copy>
- Doubling coordinates:
 - <http://tinyurl.com/DoubleCoordinates>
- Enlarging on a coordinate grid:
 - <http://tinyurl.com/Enlarge-Coord-Grid>

Download and print out the following handouts (one copy per Student Teacher, except where noted):

- ‘Exploring Similarity: Triangles and Rectangles’:
 - <http://tinyurl.com/ExploreSimilarity>
- Centimetre grid paper:
 - <http://tinyurl.com/Grid-Paper-Cm> (at least three per Student Teacher)
- ‘Using a Grid to Enlarge or Shrink a Shape’ (Detective Duck):
 - <http://tinyurl.com/Enlarge-on-Grid>
- ‘Analyzing Mathematical Instructional Tasks’ by Mary Kay Stein, Margaret Schwan Smith, Marjorie A. Henningsen, and Edward A. Silver:
 - <http://tinyurl.com/MKS-Task-Analysis>
- ‘Task Analysis Guide’:
 - <http://tinyurl.com/Task-Analysis-Guide>
- ‘Thinking through a Lesson Protocol’:
 - <http://tinyurl.com/Cog-Dem-Protocol>

Materials to copy (for Instructor's use):

- Centimetre grid paper transparencies (at least three copies):
 - <http://tinyurl.com/Grid-Paper-Cm>

Materials to bring to class:

- Lined paper
- Overhead projector
- Markers to use on transparencies
- Protractors
- Mathematics textbooks for primary and elementary grades (one per small group)

This week's maths topic addresses the geometric and visual aspects of proportion.

When Student Teachers hear the word *proportion* in a college or university mathematics course, they may think of the 'Numbers and Operations' unit, most likely because they were taught a procedure (cross-multiplying) in order to solve numerical proportions, such as $5/8 = x/56$, which would result in the equivalent fraction $35/56$ —a fraction not often found in the real world!

The use of *similarity* (geometric proportions, also called *dilations*) can help Student Teachers shift their thinking about proportions as purely numerical and procedural to a model that emphasizes visual proportions. In doing so, similarity helps develop a visual and conceptual understanding of proportionality and helps strengthen Student Teachers' understanding of numerical proportions as well.

It is important to remember that although geometric proportions are addressed formally in later grades, even young children can perceive proportionality for the objects and images in their everyday world. They understand that toys (such as a doll, truck, or plastic animal) are in proportion to these toys' real-life counterparts. They also realize that a kitten is in proportion to its mother and that a baby elephant has the same shape as an adult elephant—even if their sizes are different.

Older children understand that a map of Pakistan must be in proportion to the shape of the country's actual land area. They can also understand that when an overhead projector is used to show a transparency, the image on the screen is much larger than the original, but both the original and the projected images are in proportion to each other.

Because of their ability to perceive ‘same shape, different size’, children have an intuitive perception of two figures that are not in proportion to each other. Consider this computer image of a tiger (the original):



Figure: Original image of a tiger's face

This original image can be stretched or shrunk in various ways.

I can double the width of the original horizontally:



Figure: Original image of a tiger's face stretched horizontally

Or double the original's height vertically:



Figure: Original image of a tiger's face stretched vertically

Both of these distort the picture so that it is no longer in proportion to the original.

However, using the computer, I can push at a corner of the picture to shrink it (halving both dimensions) or to pull at a corner to enlarge it (doubling both dimensions). Either action results in the new image being in proportion to the original:



Figure: Image of a tiger's face at the original size (left) and a proportionally enlarged sized (right)

Although young children have an innate sense of proportionality, their teachers need to identify and discuss proportions that children see in their everyday world. In this way, primary grade teachers are laying groundwork for the concept of *scale factor* in the future.

The geometric use of proportion to enlarge or reduce images is referred to as *scaling*. A *scale factor* describes how much a figure is enlarged or reduced. It can be expressed as a decimal, fraction, or per cent. In the case of reducing the image on the left, the scale factor was $\frac{1}{2}$ (halving). When enlarging the image on the right, the scale factor was 2 (doubling).

Ratios and proportions are the basis for geometric dilations, where a scale factor holds the relative lengths of a figure's corresponding sides consistent while all the figure's corresponding angles remain the same.

Notice that when we halved each of the original picture's dimensions (by a scale factor of $\frac{1}{2}$), the resulting picture is only $\frac{1}{4}$ the area of the original. However, when we doubled both dimensions (a scale factor of 2), the picture is now four times the size (area) of the original. When you address these ideas to students, you will want to ask questions such as:

- Why is this so?
- What do you predict will happen to the area of the newly enlarged image if we reduce it by a scale factor of $\frac{1}{2}$?
- What would happen to the area if the original image were increased by a scale factor of 3?
- If we wanted to return this newly enlarged square (with a scale factor of 3) to its original size, what scale factor should be used? Why?
- What is the relationship between the scale factor and area? How can you prove that?
- For simple polygons, what is the relationship between scale factor and perimeter?

This week's activities address Student Teachers on their developmental continuum, offering opportunities for them to experience and experiment with geometry in a way that can help build their future students' awareness toward more abstract analysis in the future.

Also, for children whose mathematical predisposition is visual and geometric, these types of activities can build on their mathematical strength to support their understanding of proportion in the 'Numbers and Operations' unit.

This week's teaching focus (selecting mathematical tasks that have high cognitive demand) is closely related to the learning topic (maintaining the high cognitive demand of such tasks during the lesson).

Student Teachers need to understand the difference between low- and high-level cognitive demand tasks. After they have completed each of this week's activities, have them discuss how the tasks might be categorized according to Bloom's Taxonomy.

When selecting tasks, teachers need to determine if the tasks are intellectually worthwhile, mathematically significant, and worth the time that a child spends doing it. Decision-making is central to the job of teaching. Teachers need to make decisions when planning their lessons not only when selecting tasks but also in anticipating student responses. Teachers also need to make decisions during the lesson by observing how children actually respond to the planned activities and then make in-the-moment decisions to adapt their lesson to forward student learning. This is dynamic, real-time, formative assessment.

One helpful way to think about assessment is to consider several definitions of *curriculum*:

- The planned, intended curriculum
- The taught curriculum
- The learned curriculum

Ultimately, curriculum should be about what students have learned. A teacher may plan a lesson (the intended curriculum) only to discover while teaching that the lesson is not going as anticipated and needs to be adapted to meet the needs of the students. Thus, the taught curriculum differs from the intended curriculum. But education is not only about what the teacher taught. Student Teachers need to remember that student learning is the end goal.

In addition, rigour and significance must be maintained throughout a youngster's engagement with a mathematical task. The key question teachers need to ask themselves when trying to maintain rigour is, 'Who is doing most of the intellectual work?' It should be the students.

Often, if a child finds a task difficult, the teacher may feel that he or she should offer help by telling the child what to do next rather than by asking questions to access the child's prior knowledge and probe his or her problem-solving strategies. They can do so by asking questions such as:

- Do you remember doing a problem like this before?
- What have you tried already?
- What could you try next?
- Do you notice a pattern?

These kinds of probing questions ask children to think for themselves rather than rely on the teacher for procedural help. (One finding of the Trends in Mathematics and Science Studies project in the United States was that many teachers took a high cognitive level task and turned it into a low-level procedural task simply by telling students the next steps they should take to solve the problem.)

Giving students too much help can also happen in the lesson-planning stage. Although teachers need to appropriately scaffold a task so that students can succeed, over-scaffolding a high cognitive level task can reduce its learning potential. For example, in a lesson in which students are required to create a graph, a teacher might provide them with a coordinate grid that is already labelled with its title, axes, and scale rather than distributing a sheet of blank graph paper and having the students (either individually or in groups) decide on an appropriate title and scale for the data.

Week 8: Essential content and activities

Maths content

Geometric ratios

Student learning issues

Maintaining cognitive demand of mathematical tasks

Teaching the maths content

Selecting worthwhile mathematical tasks

Essential mathematical understandings

Ratios and proportions (including geometric proportions) are multiplicative, not additive.

There is a difference between the everyday use of the word *similar* and its mathematical definition.

The area of a new image increases or decreases by the square of the scale factor.

The perimeter of a new image increases or decreases by multiplying the original perimeter by the scale factor.

Except for triangles, similarity requires both corresponding side lengths and corresponding angles.

Triangles are a special case in which similarity can be determined by using only corresponding angles.

There is an essential geometric vocabulary that Student Teachers need, including terms such as corresponding, A B C (called A-prime, B-prime, C-prime), scale factor, ratio, vertex, area, perimeter, parallel, similar, and congruent.

Essential activities

Introducing similarity

Ask for examples of similarity in everyday life. Ensure that Student Teachers understand the difference between the everyday use of the word *similar* and its mathematical meaning. If no one mentions it, ask about what an overhead projector does to the original image on a transparency. Then ask about geographic maps. Do they accurately represent a city or country? Try to elicit the terms *enlarge* and *reduce*, telling Student Teachers that the word *dilation* describes changing the size of an image but not of its shape.

Have Student Teachers complete the activities on the sheet 'Exploring Similarity: Triangles and Rectangles' by drawing triangles and rectangles on lined paper. In addition to having Student Teachers discuss the questions on the handout, ask questions such as:

- How can you tell if two figures are similar?
- How would you define similarity? (Try to build a mathematically accurate definition of similarity from responses.)
- What happens to the corresponding angles when we create a similar figure?
- What happens to the corresponding side lengths when we create a similar figure?
- Why do triangles only need equal corresponding angles to ensure similarity, but rectangles need both equal corresponding angles and proportional side lengths?

Distortions and dilations

Distribute blank centimetre grid paper and ask Student Teachers to draw a square of any size, noting its side length, perimeter, and area.

Ask them to double the square's side length vertically. Then have them use the original square and double its side length horizontally. Finally, ask them to use the original square and double both dimensions. In a whole-class discussion, ask questions such as:

- What do you notice about the three new figures?
- Are any of them similar to the original square?
- What happened to the original perimeter of the square that was doubled in both dimensions?
- What happened to its area?
- If the square's side lengths were tripled in both dimensions, what would happen to the new figure's:
 - perimeter?
 - area?

Using a new sheet of grid paper, have all Student Teachers draw a 3×5 rectangle in the upper left corner. Then ask each small group to draw one of the following rectangles in the remaining space on the grid:

- A 4×6 rectangle (where 1 is added to both dimensions of the original 3×5 rectangle).
- A 5×7 rectangle (where 2 is added to both dimensions of the original 3×5 rectangle).
- A 13×15 rectangle (where 10 is added to both dimensions of the original 3×5 rectangle).
- A 6×10 rectangle (where both dimensions of the original rectangle are multiplied by 2).
- A 9×15 rectangle (where both dimensions of the original rectangle are multiplied by 3).

Have Student Teachers discuss which of these rectangles are mathematically similar to the original 3 x 5 rectangle. Ask questions to elicit the fact that similarity relates to multiplication, not addition.

At this point, introduce the term *scale factor*, noting that the word *factor* implies multiplication.

As a follow-up to working with squares and rectangles, ask how similarity works with circles, triangles, and regular polygons (one of which is a square). What generalizations can they make about these three figures? Have them explain their generalizations.

Enlarging a figure on a grid

Distribute the handout 'Using a Grid to Enlarge or Shrink a Shape'. Ask Student Teachers, working individually, to create a new figure that is mathematically similar to the one on the page. Expect Student Teachers to ask you to explain the directions but resist giving further instruction. This is where you model how to maintain the rigour of a task by not helping too much. Instead, note the procedures being used as Student Teachers engage in the task. Do some ask for rulers to measure the line segments? Are others counting lengths using the grid lines? How do they address the figure's slanted lines?

After Student Teachers finish the task, have them share their drawings and engage in a class discussion about the activity. Ask questions such as:

- What did you find difficult?
- What scale factor did you decide to use?
- Does your new figure look mathematically similar? Or does it look distorted?
- Given what you know about scale factor and area, what generalizations can you make about enlarging the original image?
- What do you think it would have been like if this activity were introduced during Session 1 this week?
- How did the activities completed earlier in the week scaffold today's task?

Finally, make a connection to the learning and teaching goals of the week, mentioning your conscious decision to let them work through this problem on their own, confident that they could solve the task without extra assistance from you.

Discussions about learning and teaching

Analysing mathematical instructional tasks

During the second session of the week, have a whole-class discussion about 'Analyzing Mathematical Instructional Tasks', the article read for homework. In addition to discussing the questions at the end of the article, ask if they think there is a difference between a task's level of cognitive demand and its difficulty.

Distribute the handout ‘Task Analysis Guide’ and ask Student Teachers to evaluate the cognitive demand of each of the activities they completed this week. Ask questions such as:

- What were the big mathematical ideas?
- Were these ideas worthwhile mathematics?
- Was the time spent on the activities worthwhile?
- Were any of the activities too procedural?
- Did they see a connection between geometric proportions and other areas of mathematics (e.g. fractions, ratios, algebra, or graphing)?

Introduce the idea of maintaining the intellectual rigour of a complex task using the handout ‘Thinking through a Lesson Protocol’. Ask questions such as:

- How could you scaffold a complex task? When is it appropriate to do so? How can over-scaffolding actually lessen opportunities to learn?
- How might you lower the cognitive demand of a task, turning a highly conceptual task into a procedural one? When is it appropriate to do so?
- How could you determine the sequence of several related tasks as a method for scaffolding while keeping the cognitive demand of each task rigorous? (Did Student Teachers notice the progression of this week’s tasks, from simple and guided to complex and open to different solutions?)
- Which should you teach first: concepts or procedures? Give a reason for your choice.

Reviewing textbooks’ mathematical tasks

Distribute the school textbooks, one per small group. Have Student Teachers look through them quickly and then select a chapter to review more thoroughly. Ask Student Teachers to address the following questions in their small groups:

- What is the overall cognitive demand of the materials?
- How would they select a high cognitive demand task from what is available?
- If a task is mainly procedural, how could they increase its cognitive demand?

Assignments

After Session 1

Have Student Teachers read ‘Analyzing Mathematical Instructional Tasks’ to prepare for a whole-class discussion.

➤ <http://tinyurl.com/MKS-Task-Analysis>

After Student Teachers have completed the ‘Using a Grid to Enlarge or Shrink a Shape’ activity:

- Have them review this PowerPoint presentation that explains how a shape can be enlarged or reduced by plotting points on a coordinate grid:
➤ <http://tinyurl.com/PPT-Dilations>
- Have Student Teachers review this website for further information about using a coordinate grid to create dilations:
➤ <http://tinyurl.com/Enlarge-Coord-Grid>

WEEK 9

Week 9: Faculty notes

Maths content

Proportional thinking

Student learning issues

Balance between concepts and skills, the role of drill and practice

Teaching the maths content

Bloom's Taxonomy of Learning applied to mathematics

Read the following articles:

- H. Wu, 'Basic Skills versus Conceptual Understanding':
➤ <http://tinyurl.com/Wu-Skills-v-Concepts>

Look through the following websites:

- 'Two Different Meanings of "More"':
➤ <http://tinyurl.com/Prop-Reas-Lesson-A>
- Absolute and relative reasoning:
➤ <http://tinyurl.com/LearnerOrg-Prop-Fract>
- 'In the Trenches: Three Teachers' Perspectives on Moving beyond the Math Wars', an article about teaching for conceptual knowledge and for procedural knowledge:
➤ <http://tinyurl.com/Con-v-Proced-1>
- 'Chapter 3: Developing Understanding in Mathematics' from *Elementary and Middle School Mathematics*, Sixth Edition, by John A. Van de Walle (Pearson 2007), about conceptual and procedural knowledge:
➤ <http://tinyurl.com/Con-v-Proced-2>

Download and print out the following handouts (one copy per Student Teacher):

- 'The Lemonade Problem':
➤ <http://tinyurl.com/Lemonade-Mixtures>
- 'Proportion Word Problems':
➤ <http://tinyurl.com/Prop-Word-Prob>
- 'Sample Number and Operations Questions Using Bloom's Taxonomy':
➤ <http://tinyurl.com/Blooms-Math-Qs>
- Bloom's Taxonomy of Educational Objectives: Math Emphasis:
➤ <http://tinyurl.com/Blooms-New-Tax-List>
 - Pictorial version at:
➤ <http://tinyurl.com/Bloom-Math-Q>

Additional resources

- J. Korth, 'Proportional Reasoning':
➤ <http://tinyurl.com/Korth-Article>

The maths topic this week is proportional thinking, a key element to help children make the transition from arithmetic to algebra. Work done in previous weeks identifying patterns and geometric proportions has set a foundation for Student Teachers to think about numeric proportions.

As Student Teachers think about proportions, they first might remember the algorithm of cross-multiplying. Again, as with other basic concepts in mathematics, it is unwise to teach this algorithm until it can be developed from what students already know about fractions, patterns, geometry, and real-life situations.

How fractions relate to ratios and proportions is interesting. For example, when children are taught to add like fractions such as $\frac{1}{4} + \frac{1}{4}$, they learn that they should add the numerators but keep the same denominator: $\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$.

However, part-to-whole ratios that 'look like fractions' are not added in the same way. Consider a basketball player who attempted four shots in yesterday's game, but only one of which scored. In today's game the same thing happened: only one of his four shots went in. What is his ratio of baskets made to attempts? One out of four yesterday, and one out of four today, for a ratio of two out of eight. Represented as fractions, 'one out of four' would be $\frac{1}{4}$. If both days' ratio of baskets made were added, it would appear that $\frac{1}{4} + \frac{1}{4} = \frac{2}{8}$ —precisely what children have been told not to do when learning to add fractions!

Another issue is that some ratios are part-to-whole whereas others are part-to-part. In the above situation with basketball, the ratio of baskets made to shots attempted is two out of eight (2:8). This is part-to-whole. On the other hand, the ratio of shots scored versus shots missed is 2:6, which is part-to-part. Note that a ratio may be written either with a fraction bar or with a colon.

Ratios as rates can be discovered by finding patterns in tables of values. For example, in the following table there is a direct relationship between the cost of a single toffee and the number of toffees bought.

Number of toffees bought	Total cost of toffees
1	Rs 6
2	Rs 12
3	Rs 18
:	:
n	$n \times 6$

This raises another important point: proportions require multiplicative, not additive, thinking. Even though the price of the next toffee can be added to the prior amount, the model is that of multiplication as repeated addition, giving rise to the general rule: $n \times 6$, where there is a factor involved. This relates to the work students did with geometric proportions last week: when beginning with a 3×5 rectangle one cannot simply add 1 to each dimension (4×6) and have a mathematically similar figure.

The reason proportions can be confusing to children is that there are so many aspects of their prior mathematical thinking that need to be refined and clarified. It is also why teachers often avoided working with these complex ideas and told their students simply to cross-multiply to find the correct numerical answer, even if children did not understand what the answer meant.

This also means that teachers need to help children understand proportions by using real-life examples, such as the following. (Note, too, the word *about* in the first two of the following statements. Often we use proportional reasoning as a tool for estimation.)

- There are *about* 20 times as many motor scooters in the airport parking lot as there are cars.
- I drove *about* 60 kilometres per hour on the highway during a 240-kilometre trip.
- I want to triple a recipe that uses one cup of rice and two cups of water.
- Three out of four dentists prefer Super-White toothpaste.

Notice that the wording of each statement asks children to think proportionally, but in different ways. Thus, while it is important for teachers to use real-life examples, they need to help children a) see different situations that require proportional thinking and b) experiment with solution strategies that will eventually lead to their understanding the short cut of cross-multiplying.

The learning concept this week challenges Student Teachers to consider an appropriate balance between concepts and skills when they begin teaching.

Many young adults were taught mathematics in a procedural manner, practicing their skills by doing many calculations using the same formula, simply with different numbers (as in using cross-multiplying to solve proportions).

The articles for class discussion were selected to provide a forum for exploring this issue. Is it concepts versus skills? Or concepts *and* skills? And how, when these Student Teachers become teachers, will they handle this dilemma?

Although the Student Teachers are probably familiar with Bloom's Taxonomy in general, the teaching component this week is about how Bloom's Taxonomy can be applied to mathematics. The taxonomy relates to this week's learning topic of procedural skills, a lower level on the taxonomy versus conceptual skills, which require higher-level thinking. The categories on the original list were knowledge, comprehension, application, analysis, synthesis, and evaluation. However, in an updated version of the original taxonomy 'A Revision of Bloom's Taxonomy: An Overview' (Krathwohl, 2001), the category names were changed to *remembering*, *understanding*, *applying*, *analysing*, *evaluating*, and *creating*, as shown in this diagram:

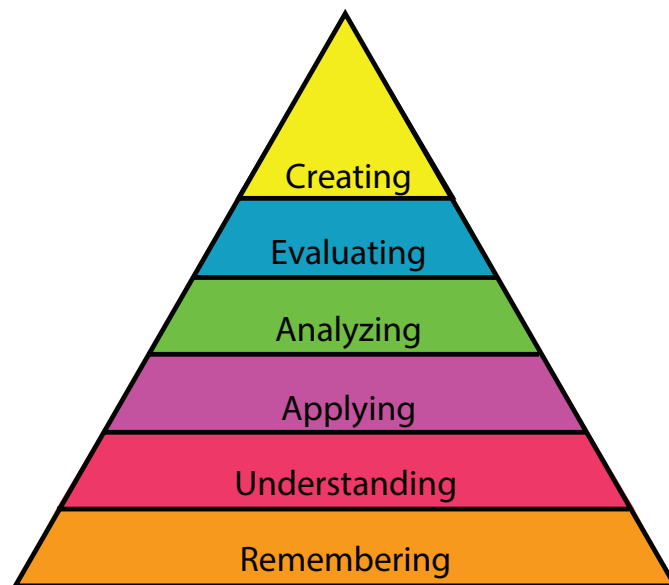


Figure: Updated version of Bloom's Taxonomy, titled 'A Revision of Bloom's Taxonomy: An Overview'

Student Teachers may assume that only older children can use higher-level thinking. Not so! For example, even small children can be asked to evaluate and justify why $2 + 2$ equals 4, while $2 + 3$ cannot equal 4.

Given this week's mathematical focus on proportional reasoning, Student Teachers grouped in pairs should be asked to do the following:

- create questions about proportional reasoning for each level of Bloom's Taxonomy
- create a proportional reasoning problem that a) has a clear real-world context for children, b) can be used in a small group discussion situation, and c) will help children move toward understanding the traditional cross-multiplying algorithm.

Week 9: Essential content and activities

Maths content

Proportional reasoning

Student learning issues

Balance between concepts and skills, the role of drill and practice

Teaching the maths content

Bloom's Taxonomy of Learning applied to mathematics

Essential mathematical understandings

Proportional thinking relates to ratios, unit rates, and per cents.

Proportional thinking is based on multiplicative thinking rather than on additive (absolute) thinking.

There is a fundamental difference between part-to-part and part-to-whole relationships.

All fractions are ratios, but not all ratios are fractions. Fractional notation can be used to write part-to-whole ratios.

All rates are ratios, but not all ratios are rates.

Rates address the relationship between two different things, such as distance (which is related to speed and time) or monetary conversion (rupees to dollars).

Introduction of the cross-multiplying algorithm to solve proportion problems should be delayed until children have engaged in activities that help them conceptualize proportional thinking.

Essential activities

Introducing proportional reasoning

The lesson is both an introduction to proportional reasoning and an assessment of Student Teachers' understanding of the concept.

Begin by asking this question from "Two Different Meanings of "More"": 'Suppose there are two classes in a school, one with 20 students and one with 25. If the first class has 10 girls and the second class has 12, which class has more girls?'

Student Teachers may quickly assume that the second class has more girls because 12 girls are more than the 10 girls in the first class. Allow enough time to have an alternative view surface. As Student Teachers discuss the problem in their groups, listen to their reasons for additive (absolute) reasoning versus proportional (multiplicative) reasoning. After they have discussed this in groups, have a class discussion about additive versus multiplicative thinking.

Developing questions with a context

Ask Student Teachers to take the question from “Two Different Meanings of “More”” and rewrite it with a local context that could make the problem more engaging to children. Ask them to compare the original and revised ways of phrasing the same mathematical question. Ask questions such as:

- Did you think both wordings of the problem were equally useful to foster understanding? If not, which version did you think was more helpful in solving the problem?
- Did the contextual phrasing give too many clues to scaffold the problem?
- Which wording may be appropriate for initial learning? Which for a unit assessment?
- How can a teacher move students from the specifics of the contextual wording to the more abstract wording?
- How is the phrasing of a question an important part of a teacher’s decision-making when planning a lesson?
- Were these conceptual or skills questions?
- Given what you know about Bloom’s Taxonomy, where would you locate these two questions?

The ‘Lemonade Problem’

Distribute the ‘Lemonade Problem’ handout, with one copy to each pair of Student Teachers. Have them work together on the problem and then share their answers with the large group. Anticipate that there may be a tendency to translate the proportions to equivalent fractions and compare the two mixtures in each problem. Note if they are using part-to-part or part-to-whole thinking, which is something to address during a whole-class discussion.

Proportion word problems

Distribute the handout ‘Proportion Word Problems’ and ask Student Teachers to extend their first answer for each problem by answering the bonus questions. These bonus questions relate to the ideas they will be exploring in Bloom’s Taxonomy. In addition, the problems can serve as models for how Student Teachers can develop word problems that are both conceptual and contextual.

Revisiting cross-multiplying

After exploring proportional reasoning, Student Teachers need to revisit their ideas about cross-multiplying as a way to solve ratio and proportion problems. Ask questions such as:

- In the problems you worked on this week, where were places you might have used cross-multiplying as a quick solution strategy?
- Now that you have explored proportional reasoning, how would you explain cross-multiplying to children in a way that is not simply procedural?

Discussions about learning and teaching

Concepts and procedures

Pose this quote from the Korth article:

... [L]earning procedural knowledge, i.e., the cross-multiply-and-divide algorithm, needs to be postponed until students have conceptualized the meaning of proportions and ratios. Yetkiner and Capraro (2009) also summarized research (Ball, 1990) that examined pre-service teachers and found that almost all of them could compute the algorithm, but only [a] few were able to develop a mathematically appropriate representation. Using multiple representations to express proportions helps students to provide powerful mental images.

How might you use this comment about research to help your Student Teachers think about the balance between concepts and procedures? Ask questions such as:

- Which methods do you think are most successful in helping students gain a good grounding in mathematics?
- What do you think causes students the most difficulty in learning mathematics?

Designing questions based on Bloom's Taxonomy

After Student Teachers have created proportional reasoning questions using Bloom's Taxonomy as assigned for homework, have them share these questions in small groups and consider which questions might be most useful for helping children in higher elementary grades understand proportional reasoning. Each group should be able to share with the class one question for each of the six Bloom's categories. Extend this to help Student Teachers realize that they can formulate a series of questions based on Bloom's Taxonomy for any mathematical topic.

Finally, note that as Student Teachers create proportional reasoning questions, they need to create proportional reasoning word problems that:

- have a clear real-world context for children
- help children conceptualize ratios, rates, and proportions
- help children better understand the procedure of cross-multiplying.

Assignments

After Session 1

Have Student Teachers read Wu's article about concepts and procedures to prepare for a discussion in the next class session.

- H. Wu, 'Basic Skills versus Conceptual Understanding':
 - <http://tinyurl.com/Wu-Skills-v-Concepts>

Have Student Teachers develop questions related to proportional reasoning using the models described in these handouts:

- Number and operations questions using Bloom's Taxonomy:
 - <http://tinyurl.com/Blooms-Math-Qs>
- Bloom's New Taxonomy list:
 - <http://tinyurl.com/Blooms-New-Tax-List>

After Session 2

Have Student Teachers review one of the following articles:

- 'In the Trenches: Three Teachers' Perspectives on Moving beyond the Math Wars', an article about teaching for conceptual knowledge and for procedural knowledge:
 - <http://tinyurl.com/Con-v-Proced-1>
- 'Chapter 3: Developing Understanding in Mathematics' about conceptual and procedural knowledge:
 - <http://tinyurl.com/Con-v-Proced-2>

These articles address questions about teaching for conceptual knowledge and understanding versus teaching for procedural knowledge and understanding. Let Student Teachers know that they will continue to discuss these ideas during the third session of the week.

WEEK 10

Week 10: Faculty notes

Maths content

Linear functions and simultaneous linear equations

Student learning issues

Multiple representations for a single mathematical concept

Teaching the maths content

Comparing models of teaching I: Deductive versus inductive, analytic versus synthetic

Look through the following websites:

- What is it? Linear equations:
 - <http://tinyurl.com/Review-Linear>
- Link sheet overview:
 - <http://tinyurl.com/Rule-of-4-Link>
- Traditional algorithms for solving simultaneous linear equations:
 - <http://tinyurl.com/Simul-by-Elimination>
 - <http://tinyurl.com/Simul-by-Substitution>
- Overview of teaching strategies:
 - <http://tinyurl.com/Teach-Strat-Overview>
- Inductive teaching 1:
 - <http://tinyurl.com/Inductive1>
- Inductive teaching 2:
 - <http://tinyurl.com/Induct2>
- Deductive teaching 1:
 - <http://tinyurl.com/Deduct1>
- Deductive teaching 2:
 - <http://tinyurl.com/Deduct2>
- Comparison of inductive and deductive methods of teaching:
 - <http://tinyurl.com/Compare-Induct-Deduct>
- Comparison of analytic and synthetic methods of teaching:
 - <http://tinyurl.com/Compare-Analytic-Synth>

Download and print out the following handouts (one copy per Student Teacher):

- Link sheet overview:
 - <http://tinyurl.com/Rule-of-4-Link>
- Water park problem:
 - <http://tinyurl.com/Link-WaterPark> (Make enough copies for half the class.)
- Pizza and movie problem:
 - <http://tinyurl.com/Link-PizzaMovie> (Make enough copies for half the class.)
- ‘Class Party Plans’:
 - <http://tinyurl.com/Link-Compare-Parties>
- ‘Simultaneous Equations’ (car and phone plans):
 - <http://tinyurl.com/Car-and-Phone-Plans>

Materials to copy (for Instructor’s use):

- Transparency of ‘Algebra Star’:
 - <http://tinyurl.com/Algebra-Star>
- Transparency of ‘Milkshakes’: Available in course resources.

Materials to bring to class:

- Graph paper so Student Teachers can create ‘link pages’
- Coloured pencils

The mathematics emphasis this week will be simple rates that can be expressed by linear equations. Thus, this week’s mathematical goals are designed to give Student Teachers a better understanding of:

- how rates can be compared in different representations
- real-life situations that can be expressed by linear equations
- what the intersection of two different linear equations means in a real-life context
- why the substitution algorithm is a short cut to find the intersection of two linear equations.

Most Student Teachers were shown how to solve linear equations by using the formula: $y = mx + b$. However, because the learning concept for this week focuses on multiple representations, Student Teachers will analyse linear equations by tables, graphs, and contextual information.

The website ‘What Is It?’ shows a variety of linear equations represented graphically. All these types of equations cannot be covered during this week. The website will, however, be helpful for you as an Instructor to assess how well Student Teachers understand some of the more basic equations.

It will be helpful to show Student Teachers the transparency of this visualization for the 'Algebra Star', which shows connections between five different ways to model the same mathematical concept:

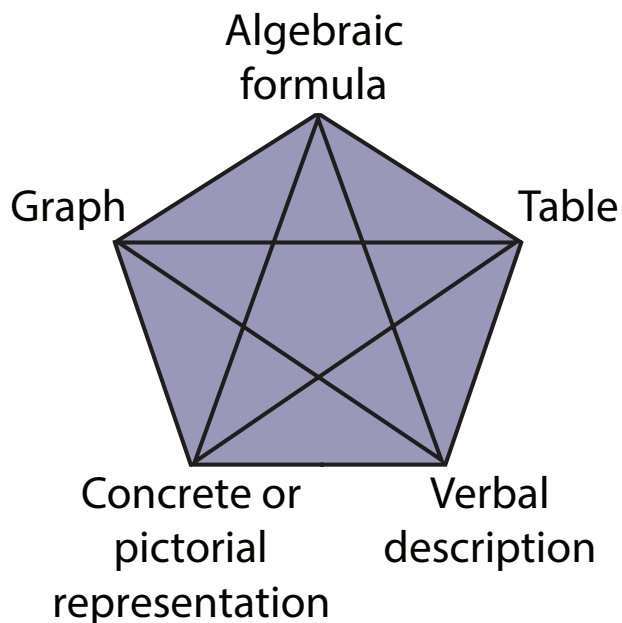


Figure: Algebra star

Make sure Student Teachers realize that the connections between the various representations do not only occur in the internal star but also between the vertices of the surrounding pentagon. And that ideally, when introducing a topic, the verbal and concrete models should precede the numerical (table), visual (graph), and symbolic (formula or equation).

During the week, Student Teachers will use a simple sheet of paper to devise 'link pages', a tool that will help them plan lessons to incorporate this important multiple representation process.

Second, most Student Teachers were taught to solve the intersection of simultaneous linear equations by subtraction or substitution procedures or both. In this course, however, the solution to the intersection of simultaneous linear equations will be introduced by graphing equations related to real-life problems to give Student Teachers a visual understanding of why the traditional algorithms of substitution and elimination work.

For the next three weeks, the instructional practices component of the course will be devoted to models of teaching. During these weeks, ask the Student Teachers to identify their own learning styles. As they listen to their peers, they should begin to think about how they would teach children who learn differently. This returns to the idea of two kinds of goals discussed in the first week of the course: teaching goals and student learning goals.

It is crucial to emphasize that student learning goals need to be first and foremost in teachers' minds. Once those goals are clear, teachers can 'work backward' to find teaching practices that answer a sequence of planning questions:

- *What* do I want my students to learn?
- *Why* do I want them to learn this concept?
- *How* can I help them learn this concept?

This week Student Teachers will compare the both the deductive and inductive and the analytic and synthetic methods of teaching. This week's emphasis builds on the work Student Teachers did last week when they explored how Bloom's Taxonomy can be used to develop questions, problems, and lesson plans.

Some Student Teachers might think of themselves as analytic; that is, working from a given algorithm, formula, or procedure that can be applied to specific situations. Others might be inclined to be synthetic, preferring to build an algorithm, formula, or procedure from real-life information.

It may be helpful for Student Teachers to make an analogy about analytic learning versus synthetic learning by thinking about another subject area: language arts. When we read and listen, we are being analytic, taking in others' thoughts and processing them in order to understand them. When we write and speak, we are being synthetic, taking many parts of what we know and putting them together to create something new. Of course, in our daily lives we move between analytic and synthetic thinking all the time. But as teachers of mathematics we need to use both systems intentionally, planning lessons and referencing the types of questions Bloom's Taxonomy suggests.

It is also interesting that some research has shown that although many teachers think analytically, moving from the whole to discover the parts, many children, especially students with special needs, see the various parts and try to put them together, almost like a jigsaw puzzle, until they can 'see' and understand the whole.

Week 10: Essential content and activities

Maths content

Linear functions, simultaneous linear equations

Student learning issues

Multiple representations for a single mathematical concept

Teaching the maths content

Comparing models of teaching I: Deductive versus inductive, analytic versus synthetic

Essential mathematical understandings

Linear functions are those that have a constant rate of change between two variables.

Continuous linear functions, when graphed, are straight lines.

Other linear equations are plotted as discrete points that imply that if these discontinuous points were connected, they would appear as a straight line. (This will usually happen on a graphing calculator.)

Linear functions can be discovered in tables and graphs and expressed by equations in the form of $y = mx + b$, where m is the slope and b is the y -intercept.

When dealing with two different linear equations (simultaneous equations), there may be a point (a,b) where the lines for each of the two equations intersect.

Essential activities

Introducing linear equations

Assess Student Teachers' prior knowledge about linear equations, asking if they remember what linear equations are and how they were taught to solve them. Also ask if they remember how to solve simultaneous linear equations. Expect that many may be vague about learning this topic. If they do remember, most likely they will describe learning to solve them by the deductive method, one of the teaching methods that will be discussed during the week. Do not attempt to review or teach the traditional algorithms or the formula $y = mx + b$ at this point! You are only trying to assess Student Teachers' current knowledge in order to guide your instruction for the week.

Let Student Teachers know that during this week they will be approaching linear equations somewhat differently. They will:

- learn to create linear equations to fit real-life situations
- learn how the same linear situation can be represented in tables, graphs, words, drawings, and sometimes manipulative materials
- experiment with methods of solving linear equations.

At this point, show the transparency 'Milk Shakes' and ask Student Teachers to work in small groups to solve the problem.

Some Student Teachers may note that there are two situations to consider and wonder how to address both of them at the same time. Rather than answer this question, ask how they might address each situation separately and then compare the results. If some Student Teachers are confused by the problem or unable to solve it, assure them that they will be exploring similar problems later in the week using a variety of strategies that should be helpful in solving these types of problems.

After Student Teachers have worked on the problem in small groups, have a whole-class discussion asking questions such as:

- Did you find a solution? What was it?
- What path did you go down first? What did you try next? Why?
- If you were not making progress, what made you recognize that your method was not leading to a solution?
- How did working with others help you in attempting to solve the problem?

It is important that Student Teachers begin to realize that getting the right answer on the first try is not how mathematicians (or children!) think, nor is it the overarching goal for this course. This offers you the opportunity to emphasize that learning from false starts has value. (This will be revisited next week when the teaching focus is on heuristics.)

Multiple representations

Introduce the learning concept of multiple representations by showing the transparency of the 'Algebra Star'. Mention that Student Teachers should think of this as the 'rule of five' when teaching any topic in mathematics. (A 'rule of four' would be more appropriate for younger children, who can create tables but not coordinate graphs.)

Tell Student Teachers they are going to solve linear functions by using a tool called a link page, which links a narrative (or diagram) to its corresponding table, graph, and equation—all on one sheet of paper.

Have Student Teachers work in pairs. Give one partner the 'Water Park' link page and the other partner the 'Pizza and Movie' link page. Explain that they are given the narrative and they now need to solve their particular problem using a table, graph, and equation (the 'rule of four.')

When they are finished, have them share their results with their partner.

Then have a whole-class discussion, beginning with the Student Teachers who solved the 'Water Park' problem. As they present their work, note any differences in the way they wrote down their responses, especially the way they scaled their graphs. Continue in a similar way, having those who completed the 'Pizza and Movie' problem present their work.

Mention that both problems involved linear equations. Ask how they learned to solve linear equations when they were in school. Were they ever asked to create an equation to model a given situation? Were they given a formula first? Did they ever use tables and graphs to find a solution? This is also an opportunity to introduce the analytic and synthetic models of teaching.

Finally, distribute a piece of graph paper to each Student Teacher. Have them fold the paper into fourths and then label each section as their link page: 'narrative or diagram', 'table', 'graph', and 'equation'.

Tell them that their assignment is to use this link page to create a narrative or diagrammatic problem that can be solved algebraically, and then to fill in the other three sections. Tell them that they will be sharing this homework assignment in the next session.

At the next class session, have several Student Teachers share the problems they created along with their link pages' table, graph, and equation.

Tell them that for homework they will need to complete the 'Class Party Plans' link sheet that asks them to compare the two party options on one sheet of paper. This assignment is intended to prepare them for the next session's focus, simultaneous equations.

Introducing simultaneous equations

The solving of simultaneous linear equations will be introduced by having Student Teachers graph equations related to one of two real-life problems: mobile phone plans or car rental plans. This focus on graphing should give Student Teachers a visual understanding of why the traditional algorithms for solving simultaneous equations (substitution and elimination) work.

Select either the mobile phone plan or the car rental comparison problem for all Student Teachers to solve. Do not mention that they are working with simultaneous equations!

Student Teachers should work in pairs, with each partner creating a link page to solve his or her part of the problem. (If Student Teachers need help, note the hints on the teacher's version of the worksheet.)

When Student Teachers have finished their link page, have them compare their graph with their partner's. Then have them overlay their partner's graph onto their own. This should result in a point where the two lines intersect.

After they have finished comparing their graphs, hold a whole-class discussion about the two different link pages and ask the following questions:

- When they compared their tables, what did they discover?
- When they combined their graphs, was there a point where the two lines intersected?
- What was this point in (a,b) format?
- What does this point mean in the context of the problem?

At this point, note that they have been working with simultaneous linear equations and that they found the solution by using their graphs. Ask how they solved these kinds of equations in the past. Then ask for volunteers who can solve the two equations using the substitution and elimination algorithms. If Student Teachers cannot remember these methods, show how the substitution and elimination methods are used. Then ask how these short cut methods relate to the work they did with tables and graphs.

Discussions about learning and teaching

Working with multiple representations

Use the link page to emphasize how multiple representations for the same concept can be displayed in one clear, convenient manner. Emphasize that this is a tool that can be used not only in the upper grades when children are able to create coordinate graphs as a visual, but also with younger children who can draw pictures to express the narrative, table, and equation.

Models of teaching

As mentioned above, when Student Teachers discuss the deductive, inductive, analytic, and synthetic methods of teaching and learning, ask them to assess their own preferred style. Have them listen to each other to become aware of those who learn differently than they do.

Assignments

After Session 1

Have Student Teachers use the link page begun in class to create a narrative or visual problem. They then should fill in the table, graph, and equation sections. Remind them that they will be sharing this homework assignment in the next session.

Have Student Teachers watch these two short videos that review the traditional algorithms for solving simultaneous linear equations:

- <http://tinyurl.com/Simul-by-Elimination>
- <http://tinyurl.com/Simul-by-Substitution>

WEEK 11

Week 11: Faculty notes

Maths content

Symmetry

Student learning issues

Mathematical learning styles and modalities

Teaching the maths content

Comparing models of teaching II: Heuristic, interactive, hands-on

Read the following articles:

- S. Cleaver, 'Hands-On Is Minds-On':
➤ <http://tinyurl.com/Teaching-HandsOn-2>
- C. Lambert, 'Twilight of the Lecture', Harvard Magazine (March–April 2012):
➤ <http://tinyurl.com/Teaching-Interactive>
- B. Thomson and J. Mascazine, 'Attending to Learning Styles in Mathematics and Science Classrooms':
➤ <http://tinyurl.com/Learn-Styles-Dunn>
- J. Y. S. Tiong, 'Top-Down Approach to Teaching Problem Solving: Heuristics in Mathematics':
➤ <http://tinyurl.com/Tiong-Heuristics>

Look through the following websites:

- Symmetry in the early grades:
➤ <http://tinyurl.com/Sym-Primary-Grades>
- Symmetry rules:
➤ <http://tinyurl.com/Sym-Rules>
- Symmetry:
➤ <http://tinyurl.com/Sym-Webquest>
- Symmetry unit (Annenberg Learner):
➤ <http://tinyurl.com/Learner-Sym-Overview>
 - Line symmetry:
➤ <http://tinyurl.com/Learner-Sym-Refl>
 - Rotation symmetry:
➤ <http://tinyurl.com/Learner-Sym-Rotate>
 - Translation symmetry:
➤ <http://tinyurl.com/Learner-Sym-Translate>
- Line symmetry (lesson ideas):
➤ <http://tinyurl.com/Illum-Lines-of-Sym>

- Symmetry activity:
 - <http://tinyurl.com/Reflection-Applet-1>
- Mirror tool (requires Java plug-in):
 - <http://tinyurl.com/Reflection-Applet-2>
- Interactive geoboard:
 - <http://tinyurl.com/Interactive-Geoboard>
- Transformed images:
 - <http://tinyurl.com/MIF-SYM-Overview>
- Reflection symmetry:
 - <http://tinyurl.com/MIF-Reflect-1>
- Folding test (diagonal of rectangle: not a line of symmetry):
 - <http://tinyurl.com/MIF-Reflect-2>
- Rotational symmetry:
 - <http://tinyurl.com/MIF-Rotate>
- Learning modalities and maths:
 - <http://tinyurl.com/Learn-Modalities>
- ‘Case for Hands-On Learning’:
 - <http://tinyurl.com/Teaching-HandsOn-1>

Download and print out the following handouts (one copy per Student Teacher except where noted):

- ‘Creating Lines of Symmetry’:
 - <http://tinyurl.com/ILines-of-Sym-Handout>
- Coordinate grid paper:
 - <http://tinyurl.com/Coord-Grid> (three copies per Student Teacher)
- ‘Pattern Blocks’:
 - <http://tinyurl.com/PatBlk-Template>
- ‘Rotation Cut-outs’:
 - <http://tinyurl.com/Rotation-Template>
- Making paper snowflakes:
 - <http://tinyurl.com/paper-snoflakes>

Materials to copy (for Instructor’s use):

- Transparency of coordinate grid paper:
 - <http://tinyurl.com/Coord-Grid>

Materials to bring to class:

- Graph paper
- Plain paper
- Tracing paper
- Scissors
- Coloured pencils or markers
- Optional: potato and/or lemon, knife, shallow dish, and paint

Just as students need to develop a variety of ‘number senses’ such as whole number sense, fraction sense, and integer sense, they also need to develop a variety of ‘geometric senses’. One of these is spatial sense, with which children learn to move shapes around—beginning with manipulatives, then with drawings (including those on coordinate grid paper for older children), and finally abstractly—to visualize in their mind how two-dimensional shapes flip, rotate, and slide around a plane.

The ideas developed in this week’s sessions on symmetry involve transformations by reflection, rotation, and translation that involve congruence. It extends a previous week’s work on geometric proportional reasoning that emphasized similarity. However, the need to consider corresponding sides and angles remains an important concept.

These three types of transformations are all actions, and each has its own way of creating congruent images:

- Reflections take place over a *line* (or *axis*) of *symmetry*.
- Rotations take place around a point called the *centre of symmetry*.
- Translations maintain their original orientation but move around the plane, on a *plane of symmetry*.

Reflection symmetry is what we see when we look in a mirror. In fact, reflections are often called ‘mirror images’. When first learning to work with reflections, young children often fold a sheet of paper in half, draw a simple design along the fold, cut the shape out, and open it up. They are enchanted to see the whole design they created. Of course, the whole was actually the original design they drew. The part on the other side of the original was a new image: a reflection.

In more formal terms, each point of a figure is equidistant from the line of symmetry to its corresponding point in the image. Students can prove this by measuring the distance between corresponding points either on folded cut-outs or on the originals and images drawn on a coordinate plane where the line of symmetry is either the x - or y -axis.

An important idea to emphasize is that although a given figure may have reflection symmetry, finding the line of symmetry is important. For example:

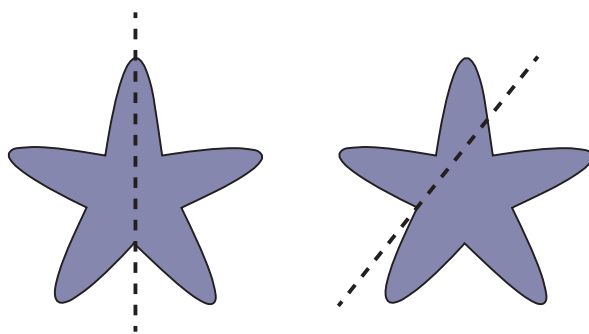


Figure: Reflection symmetry (left), not an axis of reflection symmetry (right)

Additional points you want Student Teachers to discover about reflections:

- Some figures have multiple lines of symmetry.
- All regular polygons have multiple lines of symmetry. The number of lines of symmetry is equal to the number of sides of the regular polygon.
- Circles have infinite lines of symmetry.
- Some figures, such as rectangles, may appear to have a particular line of symmetry. However, if tested by folding, it becomes clear that this is not so:

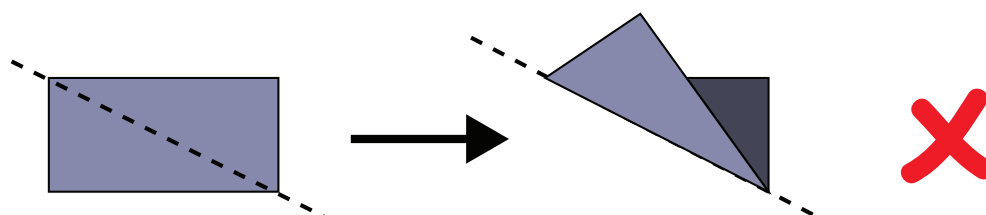


Figure: Rectangle lacks a line of symmetry

In the second session of the week, the focus is on rotation symmetry. Realize that although Student Teachers may not have learned about this topic, they probably have good intuition about the idea. But do they understand the mathematics behind their intuition? One way to assess their prior knowledge is by asking them to recall their work with tessellations in Semester 2, where they explored the fact that there are 360-degrees around a point. If the 'point' is now called the 'centre of rotation', they should be able to visualize how a shape can turn around a point one time every 360-degrees, two times every 180-degrees, four times every 90-degrees, five times every 72-degrees, six times every 60-degrees, eight times every 45-degrees, and so forth.

Rotations occur when every point of the original figure moves in a circular path around the centre of rotation. In real life, items such as blades on a fan and the wheels on a bicycle or car have their centre of rotation in the centre of the figure.

The centre of rotation also can be outside the figure. This is a more difficult concept for children to grasp. Most secondary school textbooks use geometric shapes to teach this concept, where P is the centre of rotation.

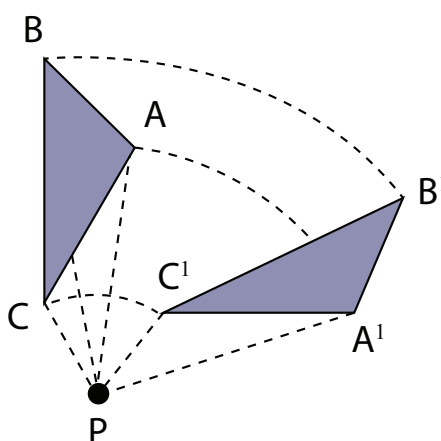


Figure: Centre of rotation outside the figure

These two pictures may help better explain this concept to Student Teachers, who will be teaching children in elementary grades.

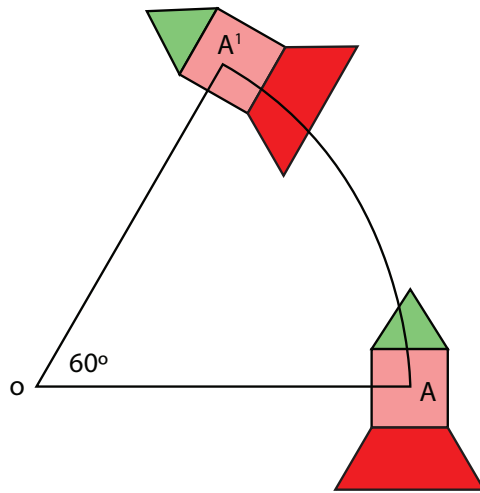


Figure: Rocket ship on a rotational symmetry of 60-degrees

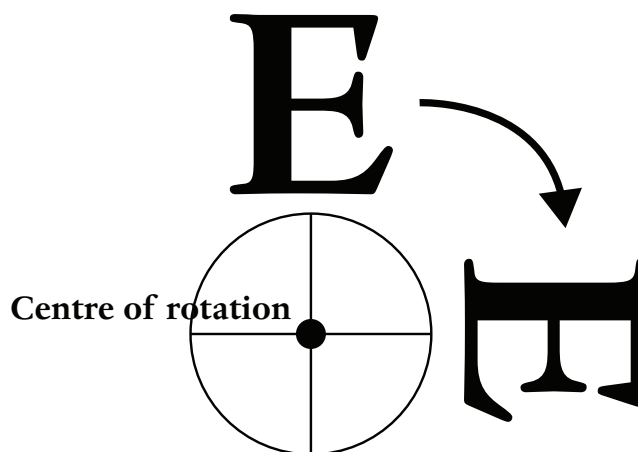


Figure: The letter 'E' on a centre of rotation angle of 90-degrees

The rocket ship has a reflection line of symmetry. But that is not the issue. We are dealing with rotational symmetry, and as indicated in the diagram above the angle of rotation is 60-degrees. This means the image will go around the centre of rotation six times in order to return to its original place. For the 'E', the angle of rotation is 90-degrees, which means the original needs to go around the centre of rotation four times for it to return to its original position.

Translations are sometimes called *slides* because they maintain their original orientation but are shifted around the plane. This can be done in any direction, not just horizontally or vertically. Mathematically, each point on the image is not only equidistant from its corresponding point on the original but also moves in the same direction. In addition, when the translation is repeated, each translation is equidistant from another.

In real life, translations can be found on wallpaper, friezes (borders), and fabric designs. They can range from the simple, as in these pattern blocks, to the quite complex, as in this carpet border:

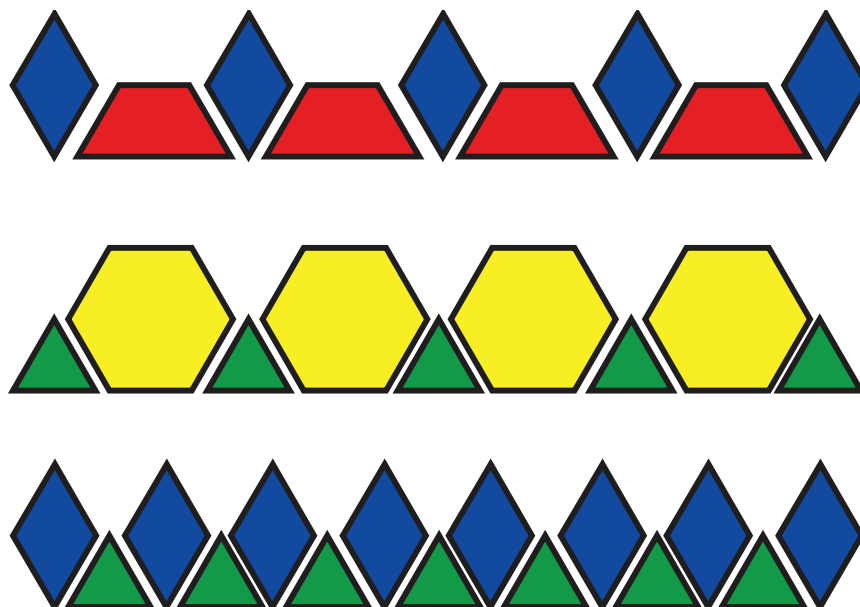


Figure: Pattern blocks exemplifying translation

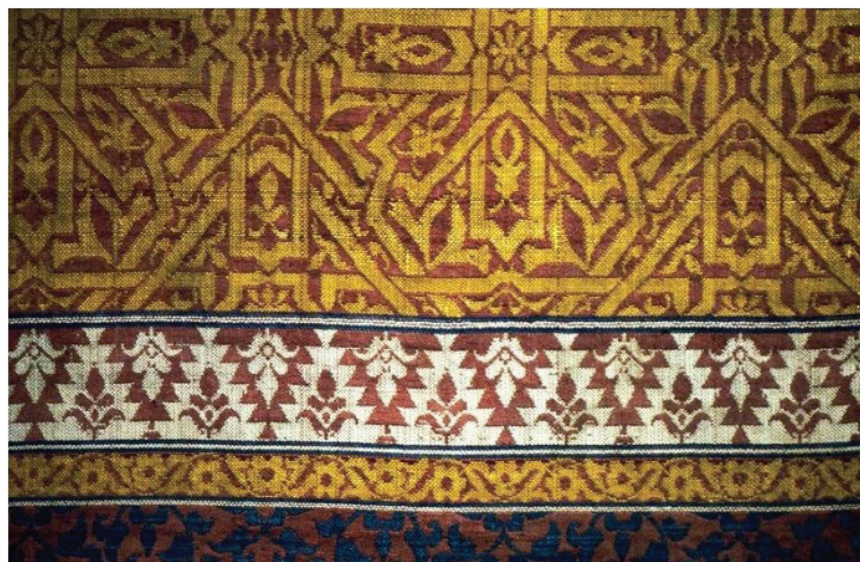


Figure: Carpet border exemplifying translation

Children can try printing congruent translations using potato stamps or bottle lids. When dipped into paint or coloured with a marker, children can create 'potato print' translations. Their translations may not be mathematically perfect, but children will be introduced to the idea of a repeated pattern, of a congruent figure, in which each image is an equal distance from each other.



Figure: Potato stamp



Figure: Potato print made from a potato stamp

Even simpler would be a 'lemon print'. This example is from a teacher in Ethiopia:

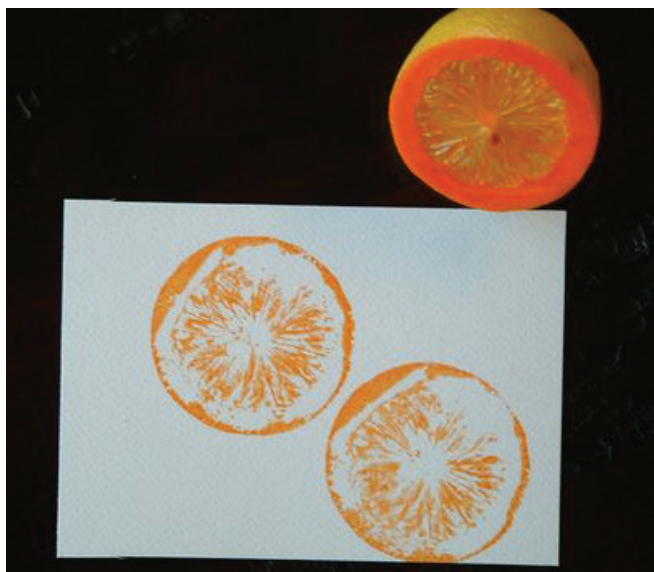


Figure: Lemon print

Transformations can also be combined, as in the picture below where the original figure (blue) on the left was reflected over the line of symmetry to the right (green), and then translated downward to its new location (red) on the plane. Student Teachers will have an activity related to combined translations in the last session of the week.

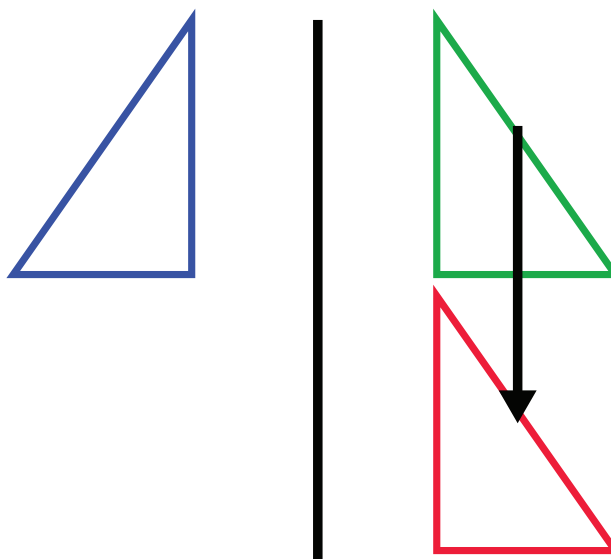


Figure: Combined translations

When working with Student Teachers, emphasize precision in vocabulary. Perhaps it is most important that they use the terms *original figure* and *image* consistently.

Student Teachers have probably encountered the learning concepts for this week (learning styles and modalities) in other courses. They should have learned that they should not view their future classes as a whole but as a collection of individual students, each of whom has unique learning needs.

The goal this week is to help them apply these general understandings to mathematics in particular. So before reviewing Student Teachers' prior knowledge about these theories, ask several important questions in the first session of the week:

- With regard to learning modalities, how do you best learn mathematics? Which do you think is your preferred processing style?
 - Visual: analysing pictures, tables, and graphs?
 - Auditory: listening to teachers' and peers' explanations?
 - Tactile/kinesthetic: manipulating materials and/or drawing pictures and diagrams?
 - Reading: reading (and re-reading) written material?
 - Writing: writing (and editing) your thoughts?

There is a second set of questions to ask that reflects back to the 'My Mathematics' assessment Student Teachers completed the first week of the course:

- When growing up, which part of maths did you find easiest and like the most?
 - Numbers and operations
 - Algebra
 - Geometry

It is important that Student Teachers do not assume that their future students think mathematically in the way they do, or in the way that textbooks are written. Although it is tempting for teachers to build on the strengths of children who are numerically inclined, it is even more important to figure out ways to teach mathematics to children who are not numerically inclined. It may be helpful to assign one of these three readings for homework and a discussion later in the week.

- 'Attending to Learning Styles in Mathematics and Science Classrooms':
 - <http://tinyurl.com/Learn-Styles-Dunn>
- Learning modalities and maths:
 - <http://tinyurl.com/Learn-Modalities>
- Multiple intelligences and maths:
 - <http://tinyurl.com/Learn-Multiple-Intell>

For this week, there are three instructional methods to discuss with Student Teachers: heuristic, interactive, and hands-on.

Heuristic refers to exploratory problem-solving techniques that we discover almost by trial and error. Prepare for class by reading the article 'Top-Down Approach to Teaching Problem Solving: Heuristics in Mathematics'. This article discusses 13 problem-solving approaches.

The article also adds two cautions:

- Heuristics do not guarantee a solution. All heuristics do is point us toward possible ways in which we might find our solution.
- Heuristics do not come with specific procedures. When using heuristics, we are required to make judgements about what we should do.

Interactive is a term that nowadays has multiple definitions. Originally it meant social interaction (as in group work) or interacting with materials (hands-on learning). Now, however, it also includes interaction with technology, as in an interactive website that allows the learner (whether adult or child) to make decisions and receive feedback. For example, there are many maths websites that are entirely text-based and therefore static. But others encourage learners to engage with the material. Here are a few interactive websites from the Mathematics I (General Mathematics) course guide to review:

- Addition of integers:
 - <http://tinyurl.com/Integer-Chip-Applet>
- Pattern blocks:
 - <http://tinyurl.com/Virt-Pat-Blocks>
- Graphing calculator:
 - <http://tinyurl.com/Free-Graph-Calc>

Student Teachers have already done a great deal of work with hands-on learning activities in both this and their previous mathematics course. To reinforce this in the week's first session, they will perform a reflection activity abstractly and then see a demonstration of a hands-on folding activity that clarifies the mathematics of reflections.

Hands-on learning can be a goal in and of itself: to solve a given problem. But it is also a way to learn a concept so thoroughly that the learner's mind is eventually trained to envision a concept even when the physical materials are not present. Thus, the goal is to use concrete models to create intellectual models that stay with a child long after the hands-on activities are over.

Week 11: Essential content and activities

Maths content

Symmetry

Student learning issues

Mathematical learning styles, modalities, intelligences

Teaching the maths content

Comparing models of teaching II: Heuristic, interactive, hands-on

Essential mathematical understandings

The symmetry of congruence includes reflections, rotations, and translations (slides).

Reflections, rotations, and translations are actions that move an original figure an equal distance to create a congruent image. This is called 'distance-preserving congruence', in which the original figure could map onto the new image if reflected, rotated, or translated.

Student Teachers can apply what they know about patterns to create transformations.

Reflections, rotations, and translations (and combinations of these) can be used to 'move' an original figure on a coordinate grid.

The vertices of new images are labelled with this symbol: '. Thus, the new vertex that would map onto the original vertex A is written A' and is called 'A-prime'.

Children can learn to recognize reflections, rotations, and translations in real-life situations such as fabric, wallpaper, architecture, and the like.

Children need to use hands-on materials such as cut-outs, manipulatives, and mirrors to discover reflections, rotations, and translations during classroom lessons.

'Anchor images' can help children remember the three different kinds of translations:

- A butterfly (reflection)

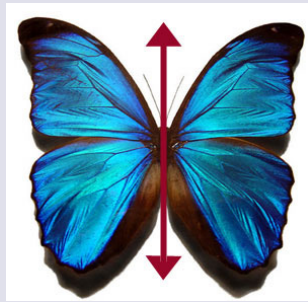


Figure: Butterfly

- A fan (rotation)



Figure: Floor fan

- A frieze, which is a repeating pattern (translation)



Figure: Carved wood panel with repeating pattern

Essential activities

Reflection (or line) symmetry

Begin this week's first session by asking Student Teachers to quickly write down the first thing that comes to mind when they hear the word *geometry*. Chart their responses to get an overview of their impressions. Common answers may be theorems, shapes, or the work they did enlarging and reducing images earlier in the course.

Next, distribute the 'Creating Lines of Symmetry' handout that asks students to identify lines of symmetry in six different figures. Have Student Teachers come to the board to sketch each figure and draw lines of symmetry. It is likely that the Student Teacher who sketches the rectangle will draw diagonals as third and fourth lines of symmetry. Ask how many agree with this. (Many assume dividing a figure into two equal parts is the same as finding a line of symmetry.)

In response to this error, demonstrate reflection symmetry by folding a sheet of copy paper, first horizontally, then vertically, and then diagonally. Student Teachers will see that with the horizontal and vertical folds, each piece 'maps' onto the other. However, the portions formed by the diagonal folds (although equal in size) do not map onto each other.

Extend the activity by asking questions such as:

- Why does a square have four lines of symmetry whereas the rectangle with different lengths and widths has only two?
- Why does a circle have infinite lines of symmetry whereas an oval has only two?
- How are the oval and rectangle alike?
- How many lines of symmetry does a scalene triangle have?
- An isosceles triangle?
- An equilateral triangle?
- What do you notice about the number of lines of symmetry in an equilateral triangle and a square? Why do you think this is so?
- Look at the regular octagon. How many lines of symmetry does it have?
- Do you notice a pattern?
- What would you predict is the number of lines of symmetry in a regular hexagon?
- Compare the lines of symmetry in a regular polygon such as a square, hexagon, and octagon. Describe how you drew them. (Both vertex to vertex and midpoint of one side to midpoint of the other side.)
- What about the lines of symmetry in an equilateral triangle and a regular pentagon? Describe how you drew them. (Vertex to midpoint of the opposite side.)
- What do you predict would be the number of lines of symmetry in an 11-sided regular polygon? What would the drawing of those lines look like?

Rotational symmetry

Have Student Teachers work in pairs for the following activities. To give them a hands-on experience with rotational symmetry, have them cut out one of each shape from the 'Pattern Blocks' handout and then colour one vertex on each. Have them trace each shape on plain paper and then begin rotating the shape with the coloured vertex over the tracing. It may be helpful to have them place a pencil point at the centre of the shape, defining the centre of rotation.

Have them create a four-column chart noting the shape, if it had rotational symmetry, and how many times it mapped onto the tracing. Have them leave the fourth column blank to add the angle of rotation. After they have explored all six shapes, ask questions such as:

- How many times did each of the pattern blocks' coloured vertices map onto the tracing? (This is called the 'order of rotational symmetry'.)
- What might they have noticed about differences between the regular polygons (square, equilateral triangle, and hexagon) and the ones that were not regular (the trapezium and two rhombuses)?
- What shapes looked the same after being turned upside down? What angle of rotation might this be?
- Which shapes looked the same after being turned quarter of the way around? What angle of rotation might this be?
- Which shape had the smallest angle of rotation?
- Which had the largest?
- How does the angle of rotation relate to a shape's order of rotational symmetry and 360-degrees?
- Why is this so?
- Can you generalize about these ideas for other regular polygons; for example, a regular octagon, decagon, or dodecagon? What about a regular n -gon?

In addition to the pattern block activity, which involves Student Teachers looking at examples of rotational symmetry, have them experiment by making paper snowflakes with the rotation symmetry handout. Ask each pair of Student Teachers to make three snowflakes of different rotational order: four, six, and eight from squares of plain copy paper. (If Student Teachers have not done this kind of paper-cutting activity before, support them with suggestions from the 'Making Paper Snowflakes' webpage.)

Ask questions similar to those above, but also:

- How did you decide on ways to fold your square to achieve different orders of rotation?
- Does your snowflake have reflection symmetry?
- If so, how many lines of symmetry are there for each of your different snowflakes? Can you draw those lines of symmetry?
- When you cut out the design for your snowflake, what was the fractional portion of the original you were cutting?
- How are both reflection and rotational symmetry shown through this activity?

Rotations and translations

The previous session's work on rotations focused on shapes with the centre of rotation inside the figure. In this session, Student Teachers will look at how the centre of rotation can be outside the figure.

Have Student Teachers work in pairs, cutting out four of the regular pentagons from the 'Rotation Cut-outs' handout. Ask them to describe the types of symmetry the pentagon has (reflection symmetry and rotational symmetry) around its centre point.

Explain that you want them to explore rotational symmetry in a different way during this session. Ask them to arrange the four cut-outs around the point (0,0) on the coordinate grid, rotating each one 90-degrees.

Ask questions such as:

- What do they notice about their design?
- Where is the centre of rotation for each pentagon?
- Where is the centre of rotation for the arrangement of the four pentagons?
- What is the angle of rotation around the arrangement's centre point?

Have them cut out four parallelograms and repeat the activity, asking:

- What types of symmetry does a parallelogram have?
- How is that the same or different from the regular pentagon?
- Where is the centre of rotation for their arrangement?
- What is the angle of rotation?
- What if they had used six shapes arranged around (0,0)? What would the angle of rotation be?

To finish the session, introduce the concept of a translation, or slide. Children can be informally introduced to translations by using an inked stamp pad and commercial stamp (or by cutting a fruit or vegetable into a stamp and dipping it into paint), then repeating the stamped design several times in a given direction on paper. If the materials are available, consider doing this activity in class.

If the materials are not available, have Student Teachers note the way the parallelograms were arranged on the original 'Rotation Cut-outs' handout, in a row of three. Explain that this was a translation or slide, and ask them to come up with their own definition of translation, which should include moving a shape in a specific direction and for a specific distance.

Have them place one shape on the coordinate grid with one of its vertices at (-1,-1). Have them place a second of the same shape with its corresponding vertex at (5,5). Ask how they would describe what they did. An informal way of describing this would be moving the shape 'six up and six over to the right'. Ask what they notice about the coordinates. How could they use what they know about integer addition to describe how the point (-1,-1) was 'translated' into (5,5)? Hopefully, they will note that they could add six to each coordinate. Ask them to consider the inverse: how would they describe translating the vertex at (5,5) to the point (-1,-1)? They should notice that they would subtract six from each coordinate. Ask them how what they did on their coordinate grid relates to their definition of moving a shape in a specific direction and for a specific distance.

Using the activities to launch discussions about learning and teaching

Models of teaching

As Student Teachers engage in the activities this week, refer to their use of interactive and hands-on learning. Ask them how or if doing what could be considered child's play—making cut-outs, folding paper, arranging designs on coordinate grids—helped them learn more about the mathematics of transformations.

Emphasize also that they were working not alone but in pairs, where verbal interaction was required to complete the task. Note that certain websites they will explore out of class are termed *interactive*. Rather than simply seeing a diagram, they will be making transformations by moving items around on their screens. Note that these websites provide simulations of the physical hands-on activities they did in class.

After they have experimented with the interactive websites, ask them to compare and evaluate their experiences. Which ones did they think modelled the direct experience best? When should a simulation be introduced? After a direct experience? As a way to introduce a topic? As more technology is introduced in the classroom, teachers need to analyse web-based simulations for their usefulness.

Finally, as they consider what they have done this week, ask how they used the concept of problem-solving in a heuristic manner. Which of the strategies did they use? How did they build on what they learned to refine their thinking?

Assignments

After Session 1

Have Student Teachers read this article about various types of symmetry:

➤ <http://tinyurl.com/Sym-Rules>

Have Student Teachers experiment with these interactive applets to explore reflection symmetry:

- Symmetry activity:
 - <http://tinyurl.com/Reflection-Applet-1>
- Mirror tool (requires Java plug-in):
 - <http://tinyurl.com/Reflection-Applet-2>
- Interactive geoboard:
 - <http://tinyurl.com/Interactive-Geoboard>

Have Student Teachers explore this website for images and explanations for many types of symmetry:

➤ <http://tinyurl.com/Sym-Webquest>

If Student Teachers need to see more examples of symmetry, assign this collection of websites that have excellent images and explanations. There are also interactive questions at the end of each page.

- Transformed images:
 - <http://tinyurl.com/MIF-SYM-Overview>
- Reflection symmetry:
 - <http://tinyurl.com/MIF-Reflect-1>
- Folding test (diagonal of rectangle: not a line of symmetry):
 - <http://tinyurl.com/MIF-Reflect-2>
- Rotational symmetry:
 - <http://tinyurl.com/MIF-Rotate>

Have Student Teachers go to the webpage '55 Interactive Teaching Tools for Mathematics' and try some of the interactive applets:

➤ <http://tinyurl.com/55-Interactive-Maths>

WEEK 12

Week 12: Faculty notes

Maths content

Volume and surface area

Student learning issues

Learning mathematics by writing

Teaching the maths content

Comparing models of teaching III: Problem-based learning, project-based learning

Read the following article:

- M. Burns, 'Writing in Math':
 - <http://tinyurl.com/M-Burns-Articles>

Look through the following websites:

- 'Designing Packages' video:
 - <http://www.mmmproject.org/dp/mainframe.htm>
- 'Building a Box':
 - <http://tinyurl.com/Build-a-Box-Lesson>
- Volume and surface area applet:
 - <http://tinyurl.com/Vol-SurfArea-App>
- Nets applet:
 - <http://tinyurl.com/CubeNets-App>
- 'Introduction to Problem-Based Learning':
 - <http://tinyurl.com/Prob-Learning>
- Project-based learning model:
 - <http://tinyurl.com/Proj-Learning>

Download and print out the following handouts (one copy per Student Teacher):

- 'Building a Box':
 - <http://tinyurl.com/Build-a-Box>
- Isometric dot paper:
 - <http://tinyurl.com/Iso-Dot-Paper>

Materials to bring to class:

- At least 24 small cubes of the same size. (These can be dice, base-10 block unit cubes, or wooden or plastic cubes.) Or, as described in the lesson plan below, a box of sugar cubes, enough so that each pair of Student Teachers will have 24 cubes to build their structures
- If you plan to do the activity with sugar cubes, you will need a knife to carefully open the sugar cube box so that it can be transformed into a 'net'.
- Plain paper
- Scissors
- Clear sticky tape

This week begins with a single surface area and volume activity that will be spread over all three sessions. Although surface area and volume were briefly addressed in the Mathematics I (General Mathematics) course, this week's activity will give an in-depth look at both topics together. Although the focus will be on a cuboid (sometimes called a rectangular prism), the same concepts can be applied to cylinders.

A second aspect of this extended lesson will be the idea of holding one aspect constant while varying another. In this case, the volume of all the cuboids created by Student Teachers will be 24 cubic units, but the different arrangements of those units will create cuboids with different surface areas.

This week's activity has been selected so as to connect maths to learning by writing and to problem- and project-based learning. This is a hands-on activity as well as an exercise in problem-solving, the organization and display of data, creating alternative formulae, and the interaction between Student Teachers working together as a class to solve a common problem. It may also serve as an example of the fact that if a formula is not given, even adults revert to the preferred approach of young children: counting.

Because the learning aspect this week is learning (or deepening one's understanding of mathematics) by writing, please leave time at the end of each session for Student Teachers to create a progress journal of what they learned both about the mathematics they were doing and the way they approached the tasks.

The teaching aspects this week are problem-based and project-based learning. Certainly this extended exploration is an example of a mini-project. In addition, it is also a problem-solving task that can benefit from reminding Student Teachers about the 13 different strategies they learned when you introduced the idea of heuristics last week. In fact, a focus on heuristics can be a way for them to consider what they will be writing in their journals at the end of each class session.

Ideally, the journal activity should be done in three parts. The first would be Student Teachers' immediate impressions at the end of the class session. Second, they should be asked to reflect further and add to their journal as a homework assignment. Finally, you will begin the next session by having them share the ideas that emerged when they wrote both in class in the previous session and at home. This final step

is important, because by referring to the text of what they wrote, those who enjoy speaking in class can be challenged to be succinct when expressing their thoughts, while those who tend to be shy or quiet will have something to share with their classmates. In other words, it is a way to achieve the goal of increased class participation.

The activity requires some preparation on your part: collect 24 small cubes of the same size or purchase a box of sugar cubes. We have found that sugar can be bought packed tightly into a box that holds 144 cubes.

If you are able to use sugar cubes, carefully cut around the edges of the box so that you can lay out the box as a net. This will help Student Teachers visualize how a two-dimensional surface can be wrapped around (or folded into) a three-dimensional object.

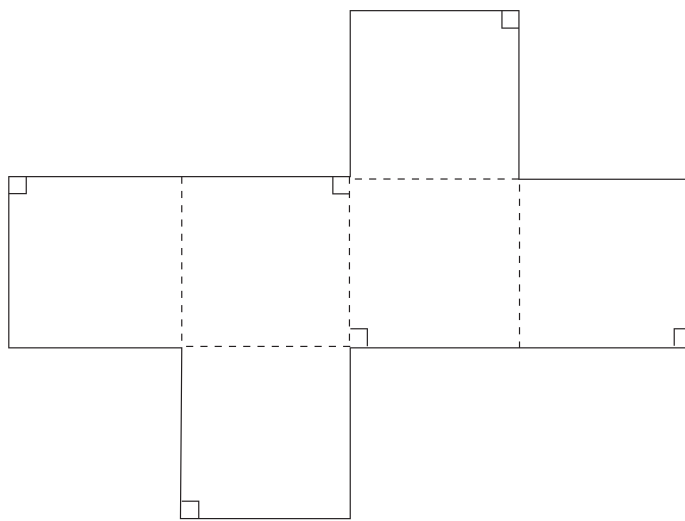


Figure: A three-dimensional box unfolded to lay flat

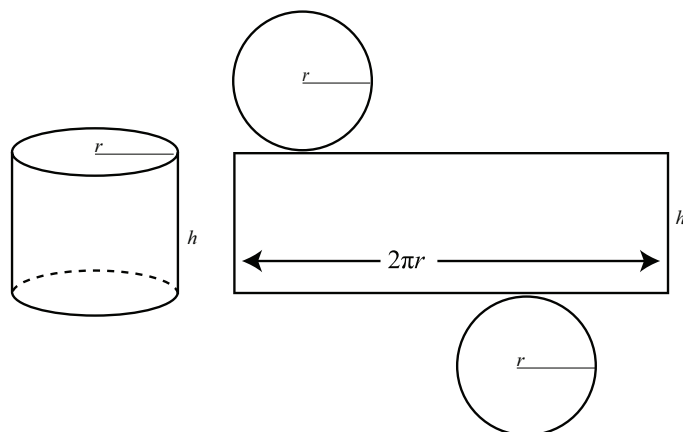


Figure: A three-dimensional cylinder unfolded to lay flat

ONE LOGISTICAL NOTE: in order for Student Teachers to build their cuboids, they need to work on a horizontal surface.

As you introduce this activity remember not to give a formula to Student Teachers, or expect them to use one. Developing a formula based on their experiences is one of the goals of this activity and something they will do in the next session.

Invite Student Teachers to build structures with a consistent 24 cubes. You can invite a pair of Student Teachers to do this while other Student Teachers make suggestions and calculate the surface area. Ask them to create a chart of their findings as follows so they can add their data as they discover it.

Volume in cubic units	Length in linear units	Breadth in linear units	Height in linear units	Surface area in square units
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Remember not to give Student Teachers a formula or to expect them to use one.

Expect that as Student Teachers suggest dimensions for their cuboids, they will probably do so in a random manner (1 x 1 x 12, then 4 x 6 x 1, etc.). This is acceptable because the second day's session will discuss why creating a second organized whole-class chart can help them discover patterns, which in turn can lead to other mathematical discoveries.

While creating their cuboids, Student Teachers may suggest designing ones with the same length, breadth, and height but of different orientations (4 x 6 x 1 versus 6 x 4 x 1). That is acceptable. If they ask about this while building, do not tell them that these are mathematically the same structures. Simply ask them to record their data on the chart.

During the second session, creating the organized chart can lead to several important mathematical discoveries.

The first will be to notice the maximum and the minimum surface area for a volume of 24 cubes in their unorganized charts. This should lead to a discussion about:

- why an organized data display can help detect patterns
- how a company wanting to make packs of 24 cubes would want to use the least packaging (surface area) to surround these 24 cubes.

If you have used sugar cubes, display the net you cut from the box of 144 cubes and ask questions such as:

- What were the length, breadth, and height of the sugar cubes' organization in this package?
- Was this packaging the most efficient way to arrange 144 sugar cubes?
- If not, was there a more efficient way to arrange these 144 cubes?

The next step would be to ask how they calculated the surface areas of their cuboids. Most will say that they counted the bottom layer (the length times breadth) and then multiplied it by the number of layers (the height).

Counting a layer and multiplying by its height is a common way to certify volume. Only then will Student Teachers begin to complete the task by counting square units of surface area.

Ask about ways other than counting to determine surface area. Different Student Teachers should discover at least three formulae for surface area. This should give rise to a discussion of equivalent expressions and how equivalence can be proved.

Further concepts about surface area and volume/capacity, noted below, should be explored during this session.

The third session this week ends with an exploration of nets and the use of isometric dot paper, which allows Student Teachers to use the data from their charts to create pictures of their three-dimensional structures on a two-dimensional sheet of paper. Be advised that some Student Teachers find this challenging, as what they know as the square sides of a cube are now being drawn as rhombuses. Rather than assisting them, have Student Teachers who are able to do this task easily help those who find it confusing and frustrating.

Week 12: Essential content and activities

Maths content

Volume and surface area

Student learning issues

Learning mathematics by writing

Teaching the maths content

Comparing models of teaching III: Problem-based learning, project-based learning

Essential mathematical understandings

We live in a three-dimensional world, where all items (even a sheet of paper) have volume/capacity and surface area.

Surface area (a two-dimensional aspect of a three-dimensional object) is measured in square units.

The volume/capacity of a three-dimensional object is measured in cubic units.

Even if an object is round or cylindrical, its surface area is measured in square units and its volume/capacity is measured in cubic units.

Surface area (a singular noun) is measured by adding all the two-dimensional *areas* of a three-dimensional object.

For cuboids, a given number of cubic units (if not a prime number) can be arranged in more than one way to create cuboids with different surface areas.

The fewest number of square units covering or enclosing a given number of cubic units will occur when the three-dimensional object approaches the square of its volume.

A net is a way to show how a three-dimensional object can be covered (or how a three-dimensional object can be enclosed).

There is a subtle difference between volume and capacity. (For example, if a plastic measuring cup with a capacity of .25 litres was mistakenly placed into the oven rather than the microwave, it would melt. This melted plastic would be the volume of the measuring cup, which no longer had capacity.)

Essential activities

Much of the lesson plan for this week has been addressed above in the 'Faculty Notes'. For additional notes, please see below.

Building cuboids

If using sugar cubes, carefully cut open the boxes of sugar cubes (as described previously) so that the boxes can be used later in the week to model a net.

Tell Student Teachers that they are going to be building cuboids from 24 small unit cubes. Ask two Student Teachers to demonstrate building various 24 unit cuboids using their classmates' suggestions. (If using sugar cubes, have them build their cuboids on a sheet of paper to collect any sugar 'dust'.)

On the whiteboard or chart paper, create a table with the headings described in 'Faculty Notes', asking Student Teachers to create a similar chart in their notebooks to record their structures' volume, dimensions, and surface area. Note that there are three types of units that Student Teachers need to consider when working on this project: a) linear (each cuboid's dimensions), b) square (surface area of the structure), and c) cubic (the constant of $24 \times 1 \times 1$ cubes).

Leave at least five minutes at the end of class for journal writing. Explain that this activity begins their emphasis on the week's learning topic: writing to learn mathematics.

For homework, assign Student Teachers to extend their initial journal entry by reflecting further on their understanding of both the mathematics of volume and surface area and the strategies they used to approach the task. Let them know that the next class session will begin with a discussion of their journal entries.

Relating volume and surface area

Begin class by having a discussion of their journal entries.

On a whiteboard or chart paper, create a table, again with the headings described above. Ask Student Teachers how they recorded their data. Continue by asking if there is a better way to organize their data in a sequential way, highlighting the value of creating an organized list in order to notice patterns.

This is also an opportunity to discuss whether the orientation of their structures (for example, $4 \times 6 \times 1$ versus $6 \times 4 \times 1$) mattered for the purposes of this activity. If so, why? If not, why not? Are there situations when a particular orientation might be important?

Once the organized list has been created, Student Teachers will probably notice the minimum and maximum surface area for a cuboid with a volume of 24 cubic units.

During this discussion, ask questions such as:

- What are the minimum and maximum surface areas for a volume of 24 cubic units?
- Why might this information be important in the real world?
- Why might the sugar cube company want to use the least packaging (surface area) to surround the 24 cubes?
- How might the concept of the minimum versus the maximum be important in other areas of mathematics?
- How can an organized data display help detect patterns that may go unnoticed in an unorganized list?

Extend the discussion by asking questions such as:

- Can you envision a prism that is not rectangular, not a cuboid? (Note that this question asks Student Teachers to consider the definition of a prism: a prism is a polyhedron consisting of two parallel, congruent faces called *bases*. If Student Teachers do not understand this definition of a prism, ask three different groups to take a sheet of paper and fold it lengthwise into either thirds, sixths, or eighths, taping the two sides together.)

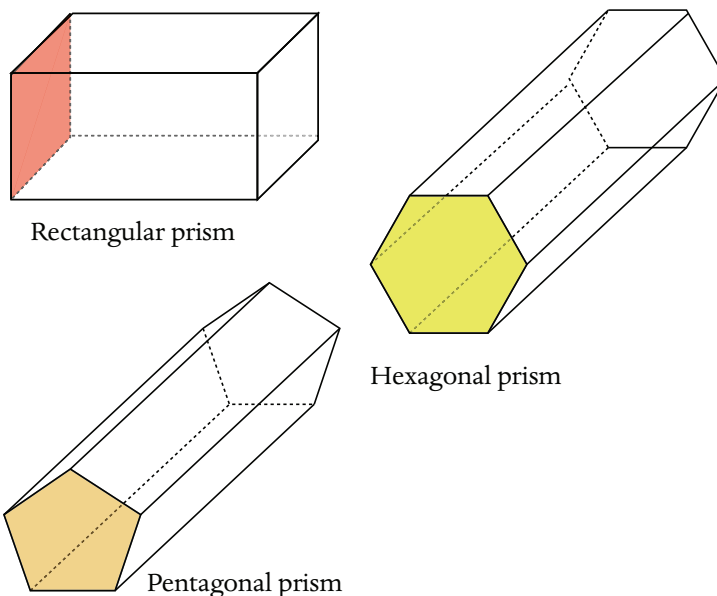


Figure: Three prisms

- What ideas do you have about finding the surface area of a prism that is not rectangular?
- How can you determine the footprint (base) of a non-rectangular prism or a cylinder?
- How can volume/capacity and surface area be calculated for a non-rectangular prism (or a cylinder) when square and cubic units cannot fit tightly into the shape or structure?

- What adaptations can be made to the specific formula for finding the surface area and volume of a cuboid (rectangular prism) in order to create a generalized formula for the volume and surface area of other three-dimensional structures?
- What is the difference between volume and capacity?

At this point, if you used sugar cubes, it would be helpful to display the net you cut from the packet, asking questions such as:

- What were the length, breadth, and height for the arrangement of sugar cubes in this packet?
- Was this packaging the most efficient way to arrange 144 sugar cubes?
- If not, was there a more efficient way to arrange these 144 cubes?

The next step would be to ask how they calculated the surface areas of their cuboids. During the demonstration, you probably saw that most Student Teachers suggested counting the bottom layer (the length x breadth) and then multiplying it by the number of layers (the height). Mention that this method of counting a layer and multiplying by the height is a common hands-on way for children to begin finding volume. Ask how did they use counting to calculate units of surface area.

Ask Student Teachers to consider that there may be other, more efficient ways to calculate surface area. What might these formulae be? There should be at least three formulae for surface area that Student Teachers suggest, such as:

- $(L \times B) + (L \times B) + (L \times H) + (L \times H) + (H \times B) + (H \times B)$
- $2(L \times B) + 2(L \times H) + 2(H \times B)$
- $2[(L \times B) + (L \times H) + (H \times B)]$

This should give rise to a discussion of equivalent expressions. Ask Student Teachers how all the surface area formulae they suggested can be proved equivalent to each other.

Allow at least five minutes at the end of class for journal writing.

Nets and isometric drawings

Begin class by having a discussion of their journal notes from the previous session, asking questions such as:

- What are they learning about the mathematics of volume and surface area?
- What are they learning about their approaches to problem-solving?
- How has recording their ideas in writing been useful in extending their learning?

The week's final two topics relate to how three-dimensional objects can be shown in a two-dimensional format. Note that these visualization tasks may be challenging for some Student Teachers.

The first part of this session is to introduce the idea of a net. If you used sugar cubes, this can be done by showing the box that you cut into a flat pattern. Can Student Teachers see how the net could be folded to recreate the original box?

Even if you did not use the sugar cube box, continue this concept by using the lesson plan for 'Building a Box', paying close attention to the section 'Questions for Students' toward the end of the lesson plan.

The second part of this session involves Student Teachers drawing one of the 24-unit structures built during the first session this week. Begin by having Student Teachers attempt to draw a cuboid in their notebook, which probably has plain, lined, or grid paper. Then give them a sheet of isometric dot paper and ask them to draw a cube at the top of the page. Ask them how they might add to this initial cube to draw at least one of the cuboids they built during the first session, such as this 4 x 3 x 2 rectangular prism:

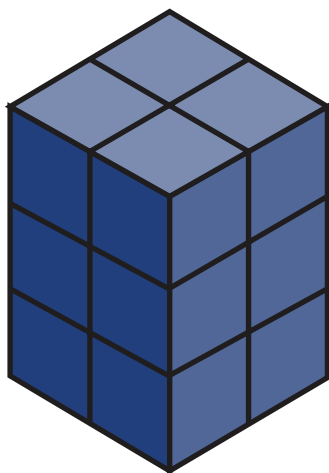


Figure: A four by three by two rectangular prism

Continue by asking questions such as:

- What shapes are you drawing to create additional cubes? Are they squares?
- Why does drawing rhombuses result in a shape that looks like a three-dimensional solid?
- Can you use isometric dot paper to draw one of the cuboids of 24 cubic units that you built earlier this week?
- Where are the 'hidden' cubes? Why can we not see them?

Discussions about learning and teaching

Models of teaching

As you move through the week, continue to reinforce the ideas of problem- and project-based learning, building on what Student Teachers had learned in previous weeks about hands-on learning and heuristics.

After Student Teachers have reviewed the websites 'Introduction to Problem-based Learning' and 'What is PBL?', engage them in a discussion of how this week's activities fit these two models of teaching and facilitation. Also ask questions such as:

- How did what you learned this week and last help you develop a collaborative mini-project for use in a practicum setting?
- At what age do you think children could begin keeping mathematics journals?
- Why might it be useful to have children add drawings and diagrams to their journal writing?
- What did you discover about yourself as a learner by engaging in project-based learning and journal writing?

Assignments

After Session 1

Have Student Teachers add to their journal reflection (after both Session 1 and Session 2) to prepare for a discussion in the following class session.

Have Student Teachers review the following interactive applet showing the relationship among nets, surface area, and volume for cuboids and cylinders:

➤ <http://tinyurl.com/Vol-SurfArea-App>

Have Student Teachers explore the following applet showing how various nets can (or cannot) be folded into cubes:

➤ <http://tinyurl.com/CubeNets-App>

Have Student Teachers review the resources on this website and choose one to share with their classmates, explaining why they thought this resource was valuable:

➤ <http://tinyurl.com/Math-Writing-Resources>

Have Student Teachers review this series of slides, 'Introduction to Problem-based Learning'. They should be prepared to discuss how this description of problem-based learning relates to this week's activities:

➤ <http://tinyurl.com/Prob-Learning>

WEEK 13

Week 13: Faculty notes

Maths content

Measurement and precision

Student learning issues

Precision in mathematical vocabulary and syntax

Teaching the maths content

Differentiated instruction

Read the following article:

- R. Capraro, M. Capraro, and W. Rupley, 'Semantics and Syntax: A Theoretical Model for How Students May Build Mathematical Mis-Understandings':
 - <http://tinyurl.com/Math-Semantics>

Look through the following websites:

- 'Measuring Accurately':
 - <http://tinyurl.com/Measure-Accur>
- 'Precision and Accuracy':
 - <http://tinyurl.com/Measure-Precis>
- 'Makeshift Measurements' lesson plan:
 - <http://tinyurl.com/Meas-MakeShift>
- 'Data Measurement and Variation' ('How long is a minute?' lesson plan):
 - <http://tinyurl.com/Measure-Minute>
- 'What is Differentiated Instruction?' (includes video):
 - <http://tinyurl.com/Intro-to-Diff>
- 'Differentiation Model' (model of differentiated learning):
 - <http://tinyurl.com/Diff-Concept-Map>
- Meeting individual learning needs:
 - <http://tinyurl.com/Diff-Instr-Math>
- Differentiating mathematics information brief:
 - <http://tinyurl.com/Math-Diff-Brief>

Download and/or print out the following handout (one copy per Student Teacher):

- 'Measurement Task Cards' available in course resources
- 'Language Difficulties in Mathematics' by Derek Haylock
 - <http://tinyurl.com/Math-Lang-1>

Materials to bring to class:

- Measuring tapes (as used by tailors)
- Scales (rulers) with inch and centimetre markings
- Metre sticks
- Graph paper
- Measuring cups used for cooking
- Scale for weighing
- Stopwatch
- Thermometer
- Paper clips to use as a nonstandard measuring unit
- Plain paper
- Scissors

This week's topic involves measurement and precision. Linking these two topics is important because students need to realize that all measurements of a continuous nature are approximations, and that increasing the degree of a measurement tool's precision allows us to obtain finer, more accurate readings.

Measurement was briefly addressed during the sessions on linear functions: that what appears to be a precise linear function for a continuous rate of change (such as driving a car at 60 km/hr) is really an estimated rate (the average of a series of speeds during a given time interval). (Along with this idea of continuous functions is that of discrete functions, in which the change can be shown by counting and the graph should be a series of separate points that are not connected.)

Another idea related to measurement that even young children can understand is that of a unit of measure. Instead of beginning with traditional measuring tools such as a tape-measure or rulers scaled with units of measurement printed on them, young children can be introduced to measurement using nonstandard units such as their foot, which is not the same as a 12-inch foot. By using their own feet to walk the length of their classroom, they will discover that those who have longer feet will walk fewer steps and those with shorter feet will walk more steps. A similar activity can be done by measuring a student desktop. If pencils of various lengths are used as a measuring tool, there will be variation in the number of 'pencil lengths' across a classroom desk. If, on the other hand, paper clips of the same size are used as a measuring tool, the number of paper clips that can be arranged end to end across the desk will be fairly consistent.

Using measures of various lengths as measuring tools provides an opportunity for the teacher to discuss what appear to be differences in the length of the room (which of course stays the same) and to enquire as why this may be so. It will also lead to asking why there might be a need for standard measurements.

Another topic related to measurement is the precision of standard measuring tools. For example, if we want to measure a pencil with a ruler that has only inch marks printed on it, children may disagree on how to read the result. Is the object being measured closer to four inches or five? How can we decide?

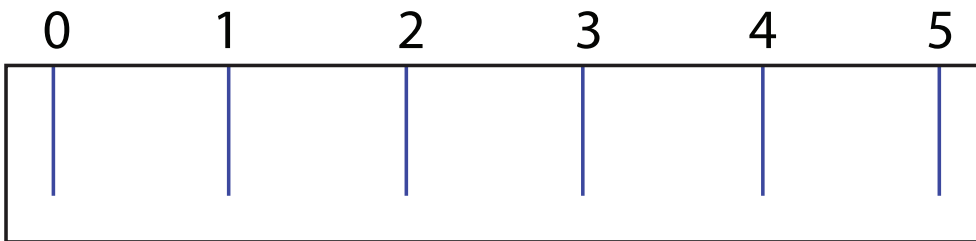


Figure: Ruler with inch marks

One way is to use a ruler with graduated markings halfway between the inch marks.

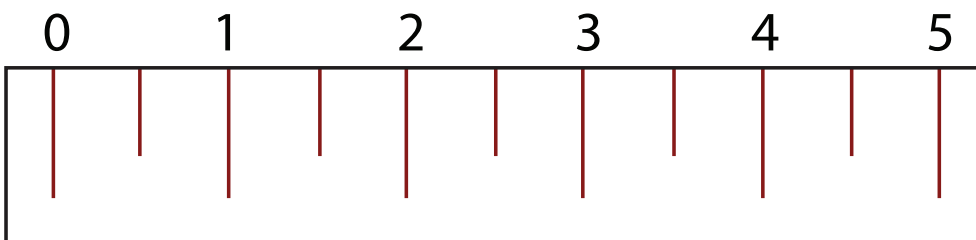


Figure: Ruler with markings in half-inch increments

If there is still disagreement, a ruler with quarter-inch markings could be used.

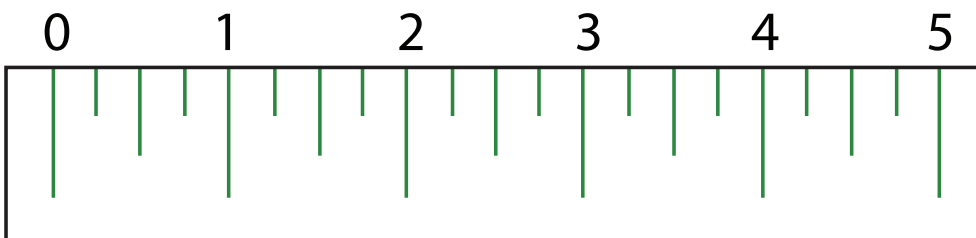


Figure: Ruler with markings in quarter-inch increments

This idea of increasing degrees of precision continues until a fairly useful approximation of length can be agreed upon.

Student Teachers may remember that this type of measurement system is actually the linear (versus the area) model for fractions. Even young children who have not yet learned about fractions can still begin thinking about what ‘three and a half’ means. In essence this is a real-world application of fractions that can pave the way to a better understanding of mixed numbers later.

Another aspect of the unit of measure is that the unit stays the same, but it is repeated. Although the examples above feature inch measurement, the unit may be a centimetre (length), a square centimetre (area), a litre (volume). The important concept is that if an object is larger than a given unit of measure, we need to use more of that same unit to find the approximate measurement.

This is not only true for solving measurement problems in the real world but also for measuring more abstract lengths, shapes, and structures.

Although these are some of the most familiar ways measurement is taught in classroom mathematics, there are other types of measurement units that are part of our lives, such as the car's odometer that tells us how far we have driven, scales that indicate how much something weighs, or clocks that measure time to varying degrees of precision (from the clock on the wall with only an hour and a minute hand, to a stopwatch used to time a race to hundredths of a second).

This week's learning concept, precision in mathematical vocabulary and syntax, complements the maths content emphasis on precision in measurement. Precision in mathematical vocabulary and syntax is developmental, and teachers play an important role in developing children's mathematical understanding by modelling correct mathematical vocabulary and syntax on an everyday basis.

When children are young they learn their basic vocabulary and syntax by listening, not by memorizing lists of words. Then, when they inevitably make a mistake, they are usually corrected by an adult modelling the appropriate word or syntax. This is why it is so crucial for primary grade teachers to know not only the informal words children use but also the more precise terms children will need in the future.

Several examples may be useful. First, if a child actually holds a ball, which is a three-dimensional object, he or she will probably say that it is round. However, when they see a picture (a two-dimensional representation) of a ball they often call it a circle, because that is what it looks like on the page. The same is true for a cube, which has square faces. Children often call a cube a 'square'. With regard to cubes and cuboids, most children refer to the faces as *sides* until their teacher mentions that the correct term is *faces*, a word children will not have heard in family conversation when they talk about the 'sides' of a box. This informal family usage versus mathematical terminology is important to address.

The second example relates to children learning their times tables in choral fashion. When students were asked to write their 'two times' table using only words, without using any numerals or symbols, they wrote what they heard: 'Two ones za two, two twos za four ...' This points to a disconnect between the non-mathematical 'za' and one of the most important conceptual words in all of mathematics: *equals*. Thus the syntax needs to be 'two times one equals two; two times two equals four'.

When adults hear the word *conversion* with regard to mathematics, they may be reminded of conversion formulae they were taught in school. But since then they probably have learned to make informal conversions. This might be called 'conversion sense'. For example, a weather forecaster might say, 'Today will be pleasantly warm with a high temperature of 20-degrees'. However, a weather website might note that the temperature will reach a high of 75-degrees. An adult with conversion sense understands these two numbers mean the same temperature and can move from one temperature measurement system to another.

A second type of conversion happens when one needs to move from one measurement unit to another. For example, if I buy a two kilogram packet of rice (weight), how many standard cooking measuring cups (volume) will the packet contain? Furthermore, how many people (a discrete number) could the whole packet (or one measuring cup) serve? People use these unit-to-unit conversions to make decisions every day. This type of conversion is far more common in daily life than using a mathematical formula to translate from one measurement system to another (e.g. metres to yards).

When working with measurement, it is important that children learn to use both the number and the units when speaking and writing. For example, when asked what they measured for a side of their classroom, many students will simply say '10'. While this numerical part of the answer may be correct, it is an incomplete answer. The full answer should be '10 metres'.

The instructional practices component for this week and next is differentiation, both in assignments and assessment. This relates to what Student Teachers have studied about learning styles. Earlier they considered the various learning styles, modalities, and intelligences that their future students will have, and which may be different from their own preferred style. But after writing about their own learning styles and challenges in their journals last week, Student Teachers should have an increased awareness of the need for teaching a given mathematical topic in ways that meet the needs of various learners. In fact, in the very first session of this course, there was this statement:

This is why it is crucial to emphasize that children's learning goals should be first and foremost in your Student Teachers' minds. Once those learning goals are clear, teachers can find teaching practices that answer the question, 'How can I help my students learn this?'

Differentiated instruction addresses the variables of content, process, products, and the learning environment in order to respond to the needs of individual learners. (This does not, however, mean that individual children need to work in an individual, isolated fashion.)

When Student Teachers engage in each activity this week, have them consider how it could be adapted for students with different learning needs. At the same time, try to model differentiation by setting up a given task in several different ways depending on what you know about Student Teachers as individuals by this point in the course. For example, how could you group them by what you know about their individual learning styles? Even if all Student Teachers do the same content activity, what different products could each group create? How could reporting and sharing their different products extend the learning of the whole class?

Alternatively, if you wanted to differentiate by considering learning environment, perhaps you might ask some Student Teachers to work individually while others work in small groups. On the other hand, might a Student Teacher who usually prefers to work individually benefit by working with just one other person rather than in a group?

Once again, the idea of differentiation is grounded in teacher decision-making based on careful observation of individual learning needs and responding to them with selected instructional practices that will foster student learning.

Week 13: Essential content and activities

Maths content

Measurement and precision

Student learning issues

Precision in mathematical vocabulary and syntax

Teaching the maths content

Differentiated instruction

Essential mathematical understandings

There is a difference between continuous quantities that are measured and discrete quantities that are counted.

All measurements of continuous quantities are approximations.

There are various units of measurement, both standard and nonstandard.

Geometric measurement can be expressed in linear, square, or cubic units.

There are other units that are used to measure weight, time, temperature, and angles.

An item can be measured by using the same unit repeatedly to reach a reasonable approximation.

There are finer degrees of measurement between two of the same unit.

Partial units of measurement can be expressed as fractions or decimals.

There are common systems of measurement such as the metric and British systems, Celsius and Fahrenheit, and the like.

We can change a measurement from one system to another by using conversion formulae.

Essential activities

Measuring with nonstandard units

The first aspect of measurement that Student Teachers will explore this week is the use of a nonstandard unit. This is often the way linear measurement is introduced in the primary grades. Approach this in two ways:

- Have some Student Teachers use multiple nonstandard units of the same size (such as paper clips) that can be laid end to end to measure the length or width of a relatively small object or surface.
- Have other Student Teachers use only one nonstandard unit, but use it repeatedly, end to end. One way to do this is to have them place a foot on plain paper, trace around their shoe, cut the drawing out, and then use it as the repeated unit to measure something such as the width of the classroom.

These activities should give rise to questions such as:

- What was the difference between these two activities?
- Could we have measured the length of the room by laying each person's individual foot cut-out end to end as we did with the paper clips? Why or why not?
- What other items could we use as nonstandard units to measure length?
- Did you have a partial unit when you were at the end of your measuring? How did you decide on what the measurement was?
- When recording or speaking about your measurement, did you note both the number and the unit name?
- Was your foot cut-out the same length as a standard 12-inch foot on a ruler?
- How accurate do you think your measurements were?
- Would using a standard measuring unit have given you a more accurate result? Why or why not?
- What do you think of the statement 'all measurements are approximations'? Do you agree or disagree?
- How could you use your nonstandard unit to measure the area of an object or shape?
- Suppose you were doing the shoe cut-out activity with young children. What general difference would there be if you also used your own shoe's cut-out to measure the side of the room? How could children benefit from this comparison?
- How would you differentiate this lesson?

Estimation, standard units, measurement sense

Although the previous activities' emphasis on nonstandard units of measurement was designed to help young children move toward the idea of standard units, the following activity allows older children who are already familiar with standard units of length and area to develop their measurement sense: estimating measurements in real-life situations by using conventionally sized objects to approximate measurements when traditional measurement tools are not available.

Begin by asking Student Teachers to envision a typical sheet of copy paper and make conjectures regarding its dimensions. How could they use their conjectures to estimate the length, width, and eventually the area of an item, such as a table top or whiteboard?

Have Student Teachers try to envision the chosen item's measurements in 'copy paper units' and then, given what they informally know about the size of copy paper, have them attempt to translate the item's dimensions from 'copy paper units' into standard units of measurement.

After they have made these estimations, have Student Teachers actually cover the selected item with sheets of copy paper. How close were their approximations in 'copy paper units'?

Finally, have them use a standard unit measuring tool, such as tape-measure or metre stick, to measure the item's dimensions in standard units. Again, how close were their approximations in standard units?

Similarly, if they knew that the span of their outstretched hand between their thumb and fifth finger is approximately 20 centimetres, how could they use this as a linear measurement tool, used repeatedly hand-over-hand, to approximate the dimensions of length and width for the item they just measured? Have Students Teachers try this.



Figure: Extend the thumb and fifth finger to use the hand as a linear measurement tool

This move from nonstandard units to standard units leads to the idea of conversion. Helping children understand that informal conversion methods (x hand spans is about the same as y centimetres) is a way to introduce the idea of formal conversion formulae.

Makeshift measurements

Use these ideas about informal and standard measurement to introduce the lesson 'Makeshift Measurements'. Please read through this lesson plan thoroughly. Make sure to click on the sections 'Questions for Students', 'Assessment Options', and especially 'Extensions' (which includes the idea of sticky notes being an easily available tool to measure area in both standard and nonstandard units).

When Student Teachers work with the 'Measurement Task Cards', the handout accompanying this lesson, they will address measurement issues involving both real-life situations and measurement sense along with both informal and standard units of measurement.

Conversion formulae

This topic addresses the idea of conversion from one system to another and from one unit to another (especially for measurements that are not expressed in linear, square, or cubic units).

Ask Student Teachers about their work with conversion formulae when they were at school. Ask questions such as:

- What do you recall, both about conversion formulae themselves and the process of learning them?
- What did you understand? What did you find confusing?
- Which of those formulae do you remember now?
- If you were shopping for vegetables, how would they convert kilograms to pounds if you could not remember the formula?
- How close an approximation of weight would you need when purchasing vegetables?
- What do you think 'conversion sense' means?
- How would you use contemporary technology to convert kilograms into pounds?

Tell Student Teachers that just as they developed formulae and algorithms for other maths topics (such as for surface area last week), they will be developing the formulae for converting temperatures from Celsius to Fahrenheit.

Have Student Teachers work in pairs for this activity. Use the following launch:

- Tell Student Teachers that this is another problem-solving activity designed to stretch their thinking about mathematics, measurement sense, formulae, and the problem-solving process.
- Activate their prior knowledge about temperature conversion formulae by asking them to recall:
 - The four measurements of degrees involved: 0-degrees, 32-degrees, 100-degrees, and 212-degrees, as well as a certain factor and its inverse
 - What these four measurements in degrees mean (the freezing and boiling points of water).

During the exploration, support Student Teachers who need additional scaffolding by asking questions such as:

- How is 0 related to 32? How is 100 related to 212?
- How can these numbers help you develop the formula?
- How might a factor or multiplier be involved? What number might that be?

Once Student Teachers have developed a formula, hold a summary discussion, asking questions such as:

- Now that you've developed the formula to convert Celsius to Fahrenheit, how could you use it to create the formula that would convert Fahrenheit to Celsius?
- What is the role of the inverse (both addition/subtraction and multiplication/division) when changing the formula?
- How can you tell if this is a linear function?
- Do you think it is discrete? Or continuous?
- How would you use your mobile phone to help you convert one temperature system to another?

More about conversion, less tangible measurements

Ask Student Teachers about conversion between units, something we do on a regular basis. If Student Teachers are confused by this concept, suggest examples such as:

- A bag of rice measured in kilograms versus the number of people that the bag of rice can serve
- Time measured in minutes versus the number of words someone can type in a given number of minutes

These examples can give rise to a discussion about rates, which Student Teachers learned about earlier in the course.

The week's final experience with measurement relates to time sense. Use the activity 'How Long Is a Minute?'

- Have two Student Teachers engage in a conversation. (This is so they do not automatically count from 1 to 60.)
- Ask each participant to indicate when he or she thinks a minute has passed.
- Record how much time actually passed.
- Repeat this activity at least two times with other pairs of Student Teachers.

After the activity, ask questions such as:

- Did the later pairs of Student Teachers get closer to the minute mark than the original pair?
- If so, why do you think that happened?
- How do you think results of this experiment may have been different if, rather than engaging in a conversation, the participants counted seconds?

Finally, ask Student Teachers how their intuitive measurement of time sense relates to their time management in daily life.

Discussions about learning and teaching

Precision in mathematical language

As you begin the week, remind Student Teachers of the learning and teaching aspects (language and syntax, and differentiated learning) that will be emphasized.

Let Student Teachers know that you will be listening especially carefully to how they 'speak mathematics' and will gently correct any errors you notice. Make sure they understand why you will be doing this, and as an assignment have them read the article 'Language Difficulties in Mathematics' after the first session. Have a whole-class discussion of the article at the beginning of the second session.

Differentiation of tasks

Also let them know that you will be experimenting with differentiation for the tasks they will be doing this week. Give a brief overview of differentiation, referring to the chart 'Model of Differentiated Learning'. Found at:

- <http://tinyurl.com/Diff-Concept-Map>

Explain that differentiation can apply to all subject areas, but that it is especially suited to the flexibility of the launch-explore-summarize lesson design model that is based on problem-solving. Ask them to be alert to opportunities in which different types of differentiation strategies could be built into this week's activities.

Language and mathematics

Begin the second session of the week with a whole-class discussion of the article 'Language Difficulties in Mathematics'. Ask Student Teachers what insights they had about children's confusion about mathematical language after reading the article. You may want to introduce the topic of children's confusion about mathematical language by having Student Teachers do the writing of times tables in words activity mentioned above. Ask how a teacher decides when to have children move to more formal or precise mathematical language.

Differentiation in mathematics

Begin the last session of the week with a whole-class discussion of the differentiated learning websites 'What is Differentiated Instruction' and 'Meeting Individual Learning Needs'. As you facilitate the discussion, ask questions such as:

- How would you adapt the National Curriculum's content to teach measurement if you had young children in your classroom who may not be ready to engage with this topic?
- How would you create scaffolding for children who appear mathematically capable but have no home experience with the topic of measurement? How can you help them learn about measurement quickly and for the first time?
- How would you create differentiated tasks for children who will be bored with your lesson because they are already comfortable with rulers and scales set to the centimetre or millimetre level?
- How do you plan for a student who struggles with the concept of conversion because he or she cannot remember conversion formulae?

Assignments

After Session 1

Have Student Teachers read the article 'Language Difficulties in Mathematics' <http://tinyurl.com/Math-Lang-1> to prepare for a class discussion.

After Session 2

Have Student Teachers review the website 'What Is Differentiated Instruction' regarding the general features of differentiated instruction. Have them also watch the short video on the website:

➤ <http://tinyurl.com/Intro-to-Diff>

Have Student Teachers review the information about 'Meeting Individual Learning Needs' that addresses how differentiated learning can be applied to early primary mathematics:

➤ <http://tinyurl.com/Diff-Instr-Math>

WEEK 14

Week 14: Faculty notes

Maths content

Estimation and large numbers

Student learning issues

Learning mathematics with available technology

Teaching the maths content

Differentiated instruction (Assessments)

Look through the following websites:

- Subitizing (Wikipedia):
➤ <http://tinyurl.com/Subitize-Wiki>
- Subitizing (Teaching Math):
➤ <http://tinyurl.com/Subitize-Resource>
- Estimation and rounding:
➤ <http://tinyurl.com/Round-to-Estimate>
- Lesson plan: large numbers, random sampling, estimation (there are five sections to this lesson):
➤ <http://tinyurl.com/Learner-Sampling>
- ‘Using Calculators in Elementary Math Teaching’:
➤ <http://tinyurl.com/Calc-Primary>
- ‘Should Kids in Primary Grades Use Computers?’
➤ <http://tinyurl.com/Computers-Primary>
- ‘Differentiated Assessment’:
➤ <http://tinyurl.com/Dif-Assess-Cda>
- Alternative assessments (NCTM):
➤ <http://tinyurl.com/Alt-Assess-NCTM>
- Balanced assessment performance tasks:
➤ <http://tinyurl.com/Balanced-Assessment>
- Mathematics Assessment Resource Service (MARS) performance tasks:
➤ <http://tinyurl.com/MARS-Assess>
- Formative assessment:
➤ <http://tinyurl.com/Exit-Qs>
- Free online graphing calculator:
➤ <http://tinyurl.com/Free-Graph-Calc>

- Wolfram MathWorld:
 - <http://mathworld.wolfram.com>
- Shodor:
 - <http://www.shodor.org>
- Cut-the-Knot:
 - <http://www.cut-the-knot.org>

Download and print out the following handouts (one copy per Student Teacher):

- ‘Instructional Technology Sampler’:
 - <http://tinyurl.com/Tech-Sampler>
- ‘Close to 100’:
 - <http://tinyurl.com/Close-100>
- ‘Methods of Alternative Assessment’:
 - <http://tinyurl.com/Meth-Alt-Assess>
- ‘Course 2: Topic Outline’:
 - <http://tinyurl.com/Course2-Topics>

Materials to copy (for Instructor’s use):

- Set of subitizing cards (print pages, cut cards out)
 - <http://tinyurl.com/Subitize-Cards>

Materials to bring to class:

- Dice, dominoes, and/or playing cards
- Basic four-function hand-held calculators, one for each pair of Student Teachers (or Student Teachers can use the calculator function on their phones)

This week will continue the emphasis on approximation and estimation, but the focus will no longer be on continuous measurement but rather discrete numbers and things that can be counted.

People in general (not just young children) first begin to estimate when they are given a set of more than about six randomly arranged items. Notice that I used the word *about* in the last sentence. This is evidence of making an estimate. For up to about six items, human beings can *subitize*, that is, intuitively and instantly determine the number of items simply by looking. (The word *subitize* comes from the Latin *subito*, which means ‘immediately’.) Once there are more than four or so items, young children start to count. But they often make mistakes by not counting all the items, or by counting some of the items multiple times. If, however, the items are arranged in a pattern, determining the number of items becomes easier. Consider how the following arrangement of tally marks is quite different from 25 ‘sticks’ scattered around the page, or how the dots on a die or domino tile can be subitized to find the sum.



Figure: 25 tally marks organised in sets of five

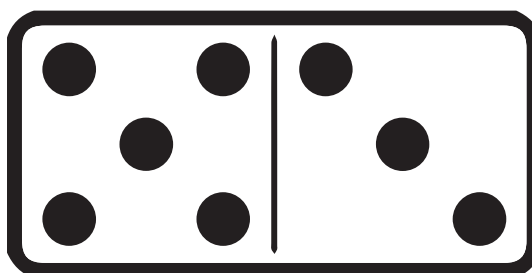


Figure: A domino piece

In fact, when children play games with dice, dominoes, and playing cards, they are not simply playing. They are learning to subitize—while practicing their addition facts.

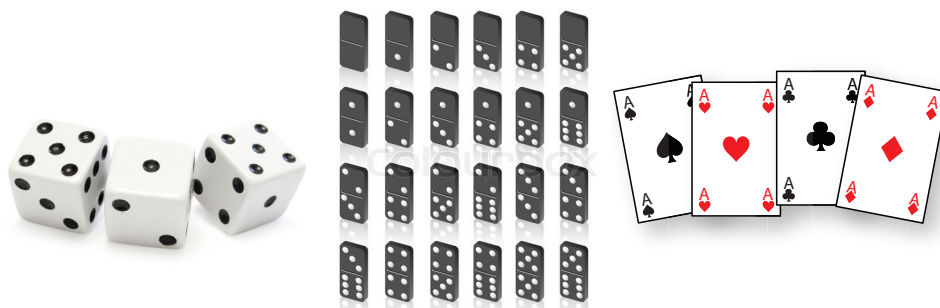


Figure: Games involving dice, dominoes, and playing cards encourage learning

However, when we are unable to count or arrange items, we instinctively resort to estimation.

Just as there are many variations on number sense (integer sense, fraction sense, decimal sense, etc.), there is estimation sense, a skill used not just for counting items but also to assess the reasonableness of an answer.

One way to think of estimation is to use the word *between*. If a youngster is multiplying 2.13 by 4.3 using the traditional paper-and-pencil algorithm, he or she may become confused as to where to place the decimal point in the answer. However, before beginning the calculation, he or she can estimate that the answer will be ‘about 8’ or ‘between 8 and 9’. Then, knowing where to place the decimal point becomes reasonable.

Even though sampling may not be included in the National Curriculum, it is still a mathematical topic that Student Teachers need to know about. Although they might have met sampling in a statistics class, it is worthwhile to briefly address it as a form of estimation used when handling large datasets in which the individual items cannot be counted accurately one by one. Instead, it involves looking at a representative portion of the entire set.

Sampling also leads to the idea of large numbers and ‘large number sense’. Fifty years ago, 1000 was a reasonable number for children to understand; thinking about someone being a millionaire was probably a mathematically intellectual stretch for children. Today, with developments in science, for example, or the increase in the world’s population, it is worthwhile to ask if children can fully comprehend powers of 10 that go far beyond their experience. And to consider that even if children can solve the calculation $47,321 \times 50,822$ on paper, their answer might be procedurally correct, but conceptually meaningless.

It is important to impress upon Student Teachers that developing estimation sense is not restricted to maths class but is a real-life skill. In everyday situations, we tend to use estimation when precision is less important and technology (the learning focus of this week) when precision is more important.

This tension between estimation and precision and knowing when which of these is better suited to a given situation is important. In a classroom, this relates to the variety of methods children need to consider when solving a problem. Teachers need to provide children with a repertoire that includes:

- Estimation
- Mental arithmetic
- Paper-and-pencil algorithms
- Hand-held calculator
- Internet and phone applications
- Computer software

For example, estimation or mental arithmetic may be the best way to determine if you have enough cash to buy your groceries. An inexpensive calculator can provide a quick answer for routine computations, such as the product of 32×3290 . When greater precision is needed for something as detailed as monetary conversions, formulae can be found by an application on your mobile phone or on the Internet. Software packages can handle complex financial information when doing your taxes.

This leads into the learning topic of the week, instructional technology. The Student Teachers have already used a variety of web-based applications to learn about maths during this course. Most of them were specific to the topic of the week. However, they should also become informed about comprehensive Internet sites that provide visuals and interactions for a wide range of topics, from simple addition to linear algebra. Some examples include Wolfram MathWorld, Shodor, and Cut-the-Knot.

Other sites have links to lesson plans that explicitly include interactive technology. But more important than simply knowing about websites is analysing their value for teaching and learning. When Student Teachers were investigating the Internet maths apps provided as links during this course, they saw a selection from literally hundreds of potential sites. Teachers, both pre-service and practicing, need to be selective when choosing resources to support student learning. They need to know that finding worthwhile apps they can use in their teaching takes time and a deep knowledge of maths concepts.

If Internet apps should be used for teaching and learning, not just practice or checking answers, so should hand-held calculators. For example, children in the primary grades can use a basic four-function calculator to play the game 'Close to 100', which allows them to work on number sense, place value, addition facts, and even integers. Older children can use the same basic calculator to work on fraction to decimal conversions, so that they begin to understand terminating or repeating decimals as rational numbers. While graphing calculators may be prohibitively expensive for individual student use, a single classroom computer can use a free online graphing calculator to generate overlapping graphs of $y = x$, $2x$, $x + 2$, and x^2 . This can be used for a whole-class discussion comparing graphs of four different functions as well as multiple representations of a table, graph, and equation.

For the second session, either arrange to have an LCD projector available or have Student Teachers bring their phones, notebooks, or laptops to class so they can explore and evaluate the websites listed in the technology sampler handout.

Last week's emphasis on differentiation focused on differentiating the task or assignment; this week the topic is differentiating the assessment.

Although traditional paper-and-pencil quizzes and standardized tests will still remain part of a child's schooling, differentiated assessments are integral to a child's education.

Look through this list of alternative ways to differentiate assessment. All these methods press for higher cognitive demand while responding to different learning needs.

- Group projects
- Journals
- Games
- Puzzles
- Oral presentations
- PowerPoint presentations
- Posters
- Drawings and diagrams
- Music and rhythm
- Physical and kinesthetic activities
- Exhibitions
- Portfolios of best work
- Portfolios showing progress
- Collections of relevant websites
- Statistical reports
- Research reports
- Observational reports
- Working with manipulatives
- Constructions
- Multiple representations
- Writing word problems
- Writing and putting on a play

But perhaps teachers' most important assessment tools are:

- observational notes taken during the exploration
- analysis notes about the summary when preparing the next day's lesson
- error pattern analysis of student work both during and after class
- rubrics that were created to clarify standards of quality work

Notice how many of these alternative assessments you have used simply because of the way this course is structured. To prepare yourself for a class discussion about differentiated assessments, take time to review the past 13 weeks and identify examples of alternative assessment models you used during the course.

Unlike paper-and-pencil tests graded from 0 to 100, these assessment methods are far more subtle and integrated into the tasks in which Student Teachers are engaged. Because of this subtlety, they may not have noticed them.

Share this list of alternative assessments with Student Teachers and ask them to recall ways, other than paper-and-pencil tests, you might have been assessing their a) progress, b) level of mathematical understanding of a specific topic, c) overall 'mathematical knowledge for teaching', and d) learning style, for example.

If you delivered Course 2 as written, hopefully the Student Teachers will begin to realize how much they learned about alternative assessment simply by being in your class.

Finally, because this is the last week that focuses on the National Curriculum's maths content, it is an opportunity for Student Teachers to reflect on their Course 2 experience. Have them use the chart of the course overview and select one or two items from each column (maths content, learning, and teaching) for an in-class written reflection. Allow time for them to review their notebooks to refresh their memories. By now Student Teachers should have become comfortable writing their thoughts and sharing these ideas with their classmates.

Before beginning the whole-class discussion, record the items from the overview that the Student Teachers selected. As you chart their responses, ask yourself:

- Are there common patterns and themes in their choices?
- How can I organize and orchestrate a discussion around any patterns or themes that I notice?

Then, after class, take time for your own reflection and ask, 'How can I use this list as a formative assessment that can support my future teaching of Course 2?'

Congratulate the Student Teachers on the work they have done during this course. But also congratulate yourself on helping these future teachers develop the 'mathematical knowledge for teaching' that will serve them (and ultimately their students) when they begin teaching.

Week 14: Essential content and activities

Maths content

Estimation and large numbers

Student learning issues

Learning mathematics with available technology

Teaching the maths content

Differentiated instruction: Assessments

Essential mathematical understandings

Estimation is a non-computational skill based on number sense.

Certain situations call for estimation, others for precision.

Estimation should be used if there is not enough information for precise computation. This would result in an answer termed ‘about ...’

Estimation can be used to determine the reasonableness of a computational answer by considering the upper and lower bounds so the answer falls between these two numbers.

Young children begin to estimate when they can no longer count with accuracy. They also estimate when rounding both whole and decimal numbers using place value concepts.

Sampling is an estimation technique used for large datasets where precise counting is not possible.

Dealing with large datasets involves both estimation and technology.

When presented with a maths problem, either in class or real life, children need to choose the most appropriate and efficient solution method: estimation, mental arithmetic, paper and pencil, calculator, or computer.

Essential activities

Estimation versus precision

In the whole-class launch, ask Student Teachers to generate ideas about how both estimation and precision are important in everyday life. Then have them work in pairs to create two real-life scenarios: the first in which estimation is reasonable, the second in which precision is required.

After they share some of these real-life scenarios, continue the discussion by enquiring when estimation could be used in purely mathematical situations, and when computational precision would be important.

Their comments can be a way to bring up the issues of using upper and lower bounds to determine the reasonable estimate of a computational answer, and rounding numbers.

Subitizing

Introduce the concept of subitizing, which may be a new idea for Student Teachers, by using the set of subitizing cards that you copied and cut apart.

Hold up a card containing seven randomly arranged dots for two seconds and ask Student Teachers to write down the number of dots on the card. Do this again with cards showing eight and nine randomly arranged dots. Make sure to do this quickly and without an explanation of why you are doing this.

Then hold up a second series of cards showing only two dots, three dots, and one dot, again giving Student Teachers only two seconds to write down the number.

Ask them how these two experiences differed. If they were unable to count the number of dots on the first set of cards, how did they arrive at an answer?

Then show cards of eight, nine, and seven dots where the dots were arranged in some sort of pattern.

After these activities, ask questions such as:

- Why was it easier to 'know without counting' the number of dots when the dots were arranged in a pattern?
- What experiences did you have as young children playing games with dice, dominoes, or playing cards that may have helped you 'know without counting'?
- How might this relate to young children's ability to perceive numbers?
- How might this relate to estimation?

Continue the discussion by noting that research shows that even infants and young children can subitize ('know the number of items without counting'). Ask Student Teachers questions such as:

- What do you think about this idea?
- How many dots do you think a toddler, a primary grade student, or an adult can know intuitively when dots are arranged randomly? (The answers: for infants three, for primary grade students five, and for adults seven.)
- Why might subitizing be an important step to help primary grade students develop number sense?
- How might subitizing relate to estimation sense, given that even most adults cannot subitize if the number of randomly arranged items is over seven?
- How might an arrangement, such as two and six on dice, a domino tile, or two playing cards, help young children learn their addition facts?

Remind Student Teachers that the initial flash card activities were designed to help them understand subitizing. They need to know, however, that when they begin teaching they can use arrangements, patterns, and games to teach children how to build both a number sense that ‘knows without counting’ as well as estimation sense when counting is not an option.

Mention, too, that subitizing also applies to the use of manipulatives in place value. When children with good organizational number sense see ‘three bundles of ten and four units’, they should be able to ‘know without counting’ that it represents 34. If, however, there are ‘eight bundles of ten and three units’, children will probably resort to estimation and suggest a total from 63 to 93. Note that they have subitized the three units but are estimating the number of tens.

Subitizing and estimating sets of items less than 10 will move to a very different level when considering large number sets that cannot be counted accurately. To prepare for the next session’s discussion of this, Student Teachers should review the website on random sampling at:

➤ <http://tinyurl.com/Learner-Sampling>

Sampling and large numbers

For homework, Student Teachers should have read through the website ‘Random Samples’ prior to class.

Ask about their experience with sampling before they read through this homework assignment. Were they familiar with the concept or was this a new idea? Explain that your goal is for them to briefly review the ‘Random Samples’ lesson plan and then ask each other questions about sampling in their small groups, and finally bring those questions to a whole-class discussion.

Mention that they have just engaged in a teaching method called ‘flipped learning’, in which they were responsible for working with content before class so that class time could be used not to introduce the material but to discuss it and clarify any confusion.

Close to 100

Begin the next segment, on instructional technology, by asking Student Teachers their thoughts about primary grade students’ use of a basic hand-held four-function calculator.

Distribute the handout explaining the game 'Close to 100'. Have Student Teachers play the game with a partner, giving each partner a different set of four digits for the first round.

- Partner 1 receives 4, 5, 9, 1.
- Partner 2 receives 3, 7, 2, 0.

For the first round of the game, they should use only paper-and-pencil record-keeping.

Have them play a second round of the game with another set of digits, only this time have them use a calculator to handle the addition of two-digit numbers.

- Partner 1 receives 8, 0, 3, 0.
- Partner 2 receives 1, 4, 7, 9.

Give Student Teachers time to talk with their partners about the discussion questions on the 'Close to 100' handout.

Bring the class together to discuss levels of intellectual activity and the use of technology when playing 'Close to 100'.

In addition to asking the questions on the handout, ask questions such as:

- Were you surprised to learn that this game was designed for primary grade children?
- What value do you think this game has for primary grade children?
- What do you think the goal is for young children who are playing this game?
 - Accurate calculation?
 - Verifying their paper-and-pencil calculations?
 - Using and building number sense to make conjectures?
- How can children with unique learning needs, who cannot yet manually add two two-digit numbers on paper using the traditional algorithm, benefit from playing this game?
- How might technology keep children engaged in the rhythm of playing the game in order to acquire understanding and insight about place value?
- How do these questions apply to children in upper elementary grades creating graphs manually versus creating those same graphs using an online graphing calculator?

Instructional technology

Now that Student Teachers have read through the two assigned websites about this issue continue the discussion begun earlier about the merits of children's use of calculators and computers. What are their thoughts?

Using the 'Instructional Technology Sampler' sheet, ask Student Teachers to review the Internet resources listed on the handout. Before they begin, ask them to consider this statement from the United States National Council of Teachers of Mathematics:

'The evaluation of materials for mathematics teaching should be an essential aspect of programme planning'.

Mention to Student Teachers that teachers need to evaluate not only their students and their own practice, but also the appropriateness and quality of the materials they use. This is especially true for Internet resources.

As they go to the resources on the 'Instructional Technology Sampler' handout, ask them to think of the criteria they are using when evaluating each for usefulness and quality.

Given that you will want to save time for Student Teacher reflection on the course, divide the sites on the handout among groups and have them report briefly.

Course reflection

Use the course outline as a journal prompt, having Student Teachers choose one or two items from each column that they felt were important in their own learning. Have them write about these and then, given the time remaining, either share in their small groups, engage in a whole-class discussion, or postpone the discussion until Week 15. However, before they leave class, be sure to take a poll of the items they thought were important to their own learning so that you can use the poll as a formative assessment about student needs, which can inform your future teaching.

Discussions about learning and teaching

Differentiation and adapted assessments

Adaptations such as giving some children more time to finish a problem set or giving other students different levels of problems, is a type of differentiation that relates to the assignment, not the assessment.

To use differentiated assessments in their future work, Student Teachers need to know about alternative assessments. This is why it is important for Student Teachers to consider alternative ways of assessing children's learning, especially for performance tasks (many of which were used during this course).

Begin a discussion by asking questions such as:

- What do you consider alternative assessments?
- How would you describe performance tasks?
- How could you create alternative assessments for the same performance task?
- What differentiated assessments might you suggest for some of the activities we have done in the past? For the activities we did this week?
- Is there a difference between evaluation and assessment? If so, how would you describe that difference?
- How might you create differentiated assessments for an activity we did this week that suits:
 - the task?
 - individual student needs?
 - the realities of time, classroom management, and available technology?
- Consider this quote: ‘The goal is not to have an individual assessment plan for each student, but to have a manageable class assessment plan that is flexible enough to accommodate a range of student needs’ (from ‘Differentiated Assessment’, Alberta, Canada). What are your thoughts about this comment?
- What is the difference between assessment:
 - of instruction?
 - for instruction?
 - as instruction?

Assignments

After Session 1

Have Student Teachers review the following two website articles:

- Subitizing (Wikipedia):
 - <http://tinyurl.com/Subitize-Wiki>
- Subitizing (Teaching Math):
 - <http://tinyurl.com/Subitize-Resource>

Have Student Teachers read through the entire lesson plan ‘Random Sampling’ at <http://tinyurl.com/Learner-Sampling>. There are several sections to this lesson. They should take notes on what they find confusing so they can ask questions during the next session.

After Session 2

Have Student Teachers review these two websites to prepare for a class discussion about the use of calculators and computers with young children:

- ‘Using Calculators in Elementary Math Teaching’:
 - <http://tinyurl.com/Calc-Primary>
- ‘Should Kids in Primary Grades Use Computers?’
 - <http://tinyurl.com/Computers-Primary>

Resources

Week 1, handout

Mathematical reflections: My mathematics

What is your earliest mathematical memory?

How did you learn your multiplication facts?

Did you experience a learning difference between your primary school maths (which probably focused on numbers and operations) and your secondary school maths (which probably began with algebra)?

Then, did you experience a learning difference when you moved from algebra to a geometry course? Which course was easier for you? Why do you think that was?

Which maths concepts were the easiest for you to learn?

Which ones seemed the most difficult?

Did you ever reach a point when you felt that maths was too difficult for you? What do you remember about that course, its teacher, and any testing or assessment related to it?

Is there any concept in maths that you still don't feel you really understand, even if you could solve problems by going through procedures? What is it?

What thoughts do you have about yourself as a learner of mathematics at this point in your life?

Week 1, handout

Analysing moves for the 1 through 30 factor game



My first move	I got this number	My opponent gets these factors	My opponent's 'factor sum'	Did I do better?
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				
26				
27				
28				
29				
30				

Week 1, handout

How many rectangles?

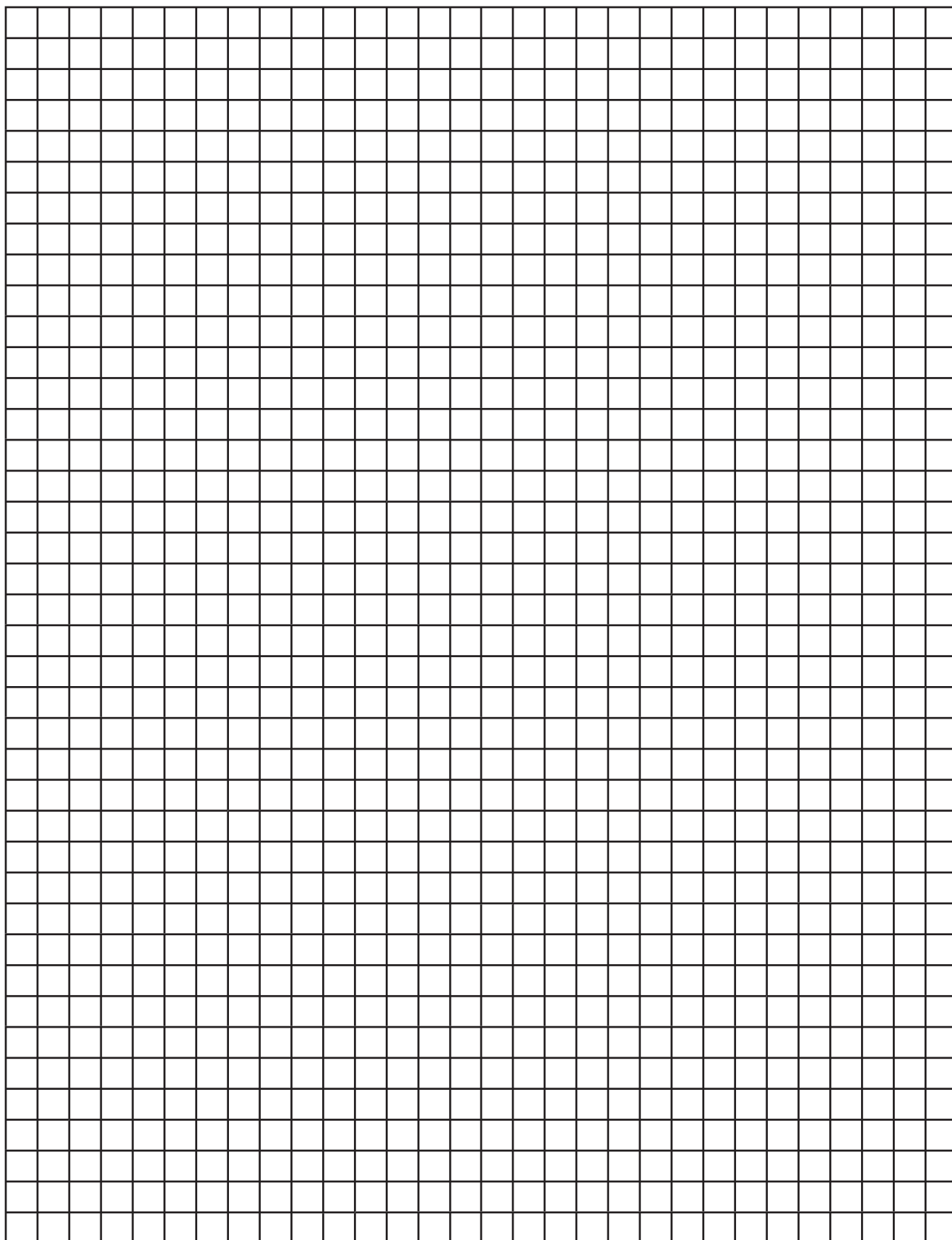


Number	Number of rectangles	Prime or composite?	Number	Number of rectangles	Prime or composite?
1			16		
2			17		
3			18		
4			19		
5			20		
6			21		
7			22		
8			23		
9			24		
10			25		
11			26		
12			27		
13			28		
14			29		
15			30		

Week 1, handout



30 x 40 grid paper



Week 3, handout

Error analysis: Munira's work samples



Review Munira's work. First, calculate the correct answer. Then ask:

- What mistakes is Munira making?
- What does she understand?
- What concepts does she not understand?
- How could I address Munira's misconceptions without simply giving her a procedure to follow?

A.

$$\begin{array}{r} 83 \\ + 67 \\ \hline 1410 \end{array} \quad \begin{array}{r} 66 \\ + 29 \\ \hline 815 \end{array}$$

B.

$$\begin{array}{r} 67 \\ + 31 \\ \hline 17 \end{array} \quad \begin{array}{r} 58 \\ + 12 \\ \hline 16 \end{array}$$

C.

$$\begin{array}{r} 24 \\ 476 \\ + 851 \\ \hline 1111 \end{array} \quad \begin{array}{r} 38 \\ 763 \\ + 693 \\ \hline 1114 \end{array}$$

D.

$$\begin{array}{r} 672 \\ - 486 \\ \hline 261 \end{array} \quad \begin{array}{r} 861 \\ - 489 \\ \hline 428 \end{array}$$

E.

$$\begin{array}{r} 61 \\ 175 \\ - 54 \\ \hline 1111 \end{array} \quad \begin{array}{r} 71 \\ 185 \\ - 22 \\ \hline 1513 \end{array}$$

F.

$$\begin{array}{r} 511 \\ 632 \\ - 147 \\ \hline 495 \end{array} \quad \begin{array}{r} 411 \\ 523 \\ - 366 \\ \hline 167 \end{array} \quad \begin{array}{r} 411 \\ 563 \\ - 382 \\ \hline 181 \end{array}$$

G.

$$\begin{array}{r} 2 \\ 17 \\ \times 4 \\ \hline 128 \end{array} \quad \begin{array}{r} 4 \\ 46 \\ \times 8 \\ \hline 648 \end{array}$$

H.

$$\begin{array}{r} 21 \\ 6 \overline{)1206} \\ \underline{1200} \\ 6 \\ \underline{6} \end{array}$$

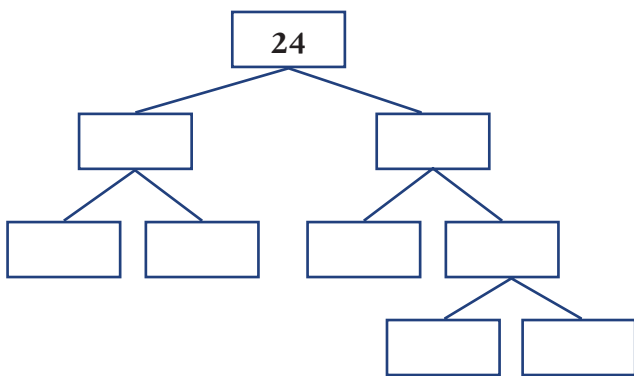
Week 3, worksheet

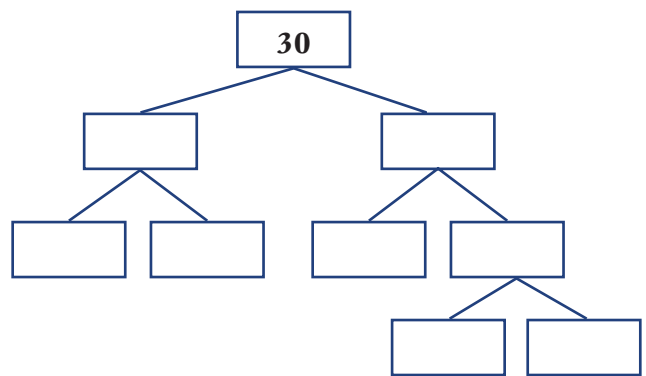
Student worksheet: Factor trees

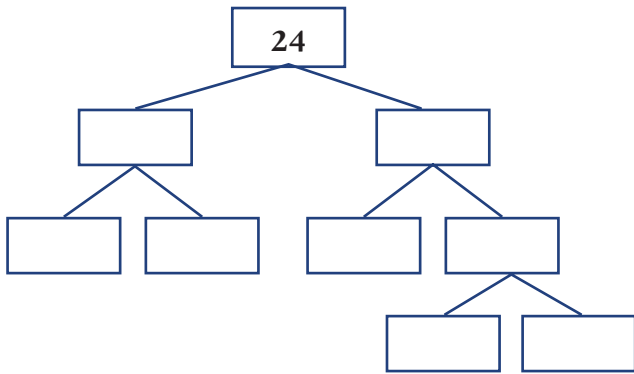


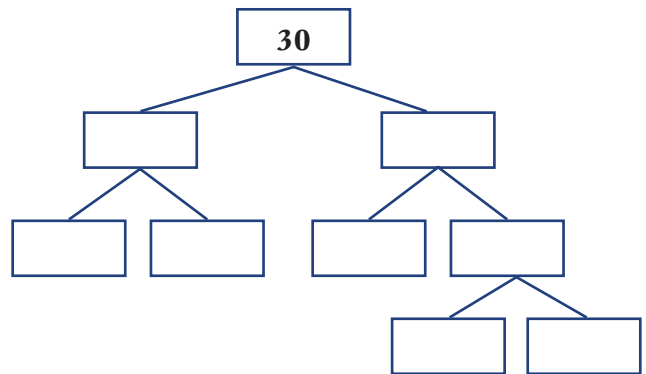
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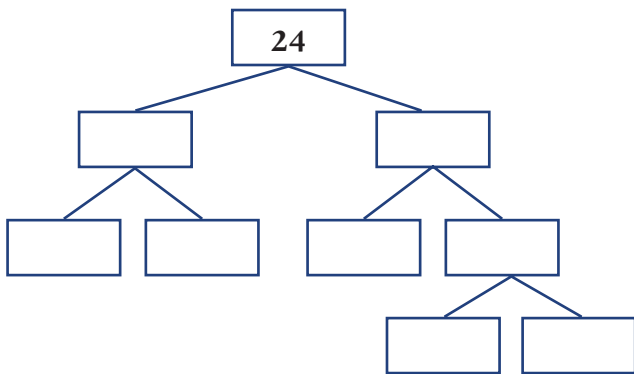
Factor each of the following numbers. If you are asked to create more than one factor tree for a number, choose different factors to begin your tree with. Color or circle the last number on each branch of each tree. Then copy those numbers into blank spaces below each tree.

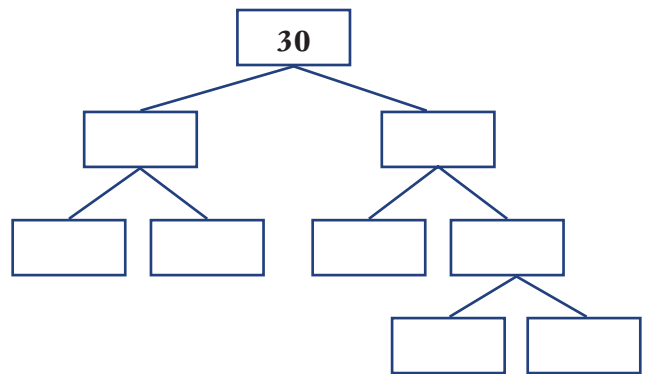












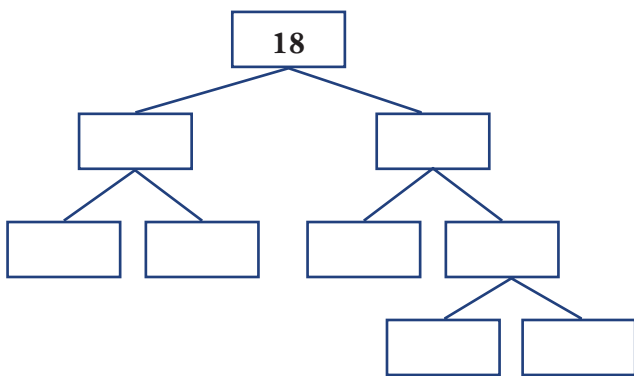
Week 3, worksheet

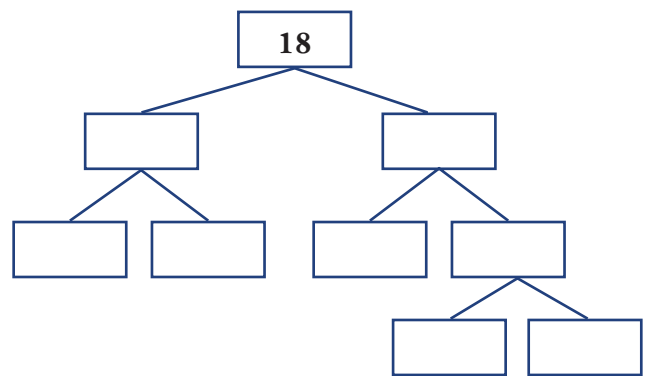
Student worksheet: Factor trees

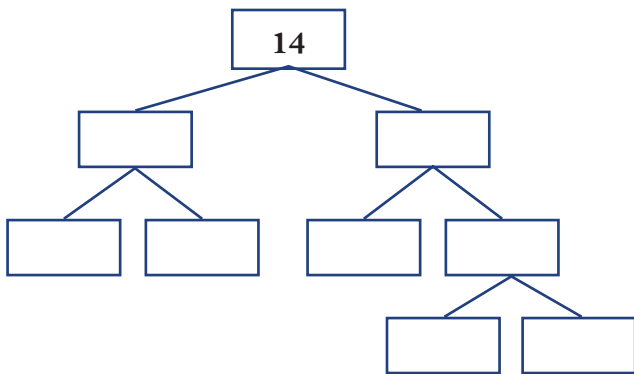
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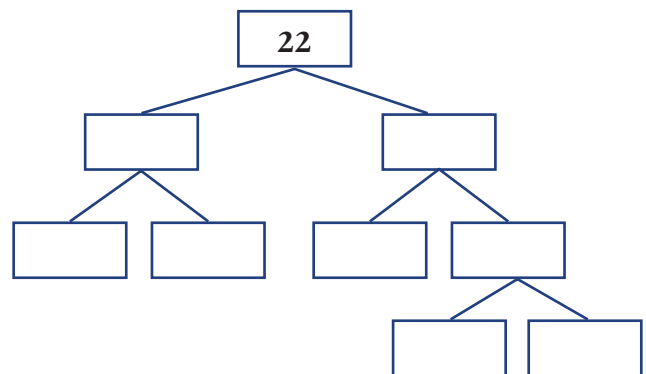
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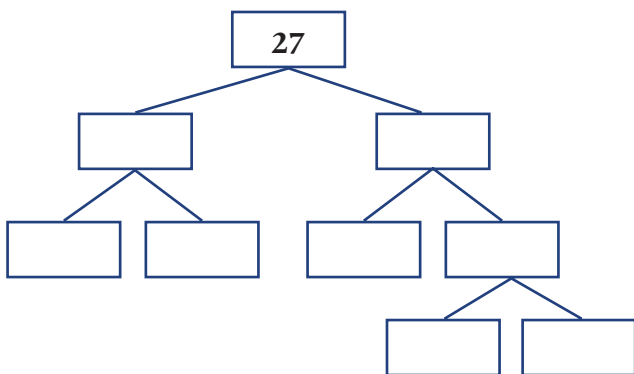
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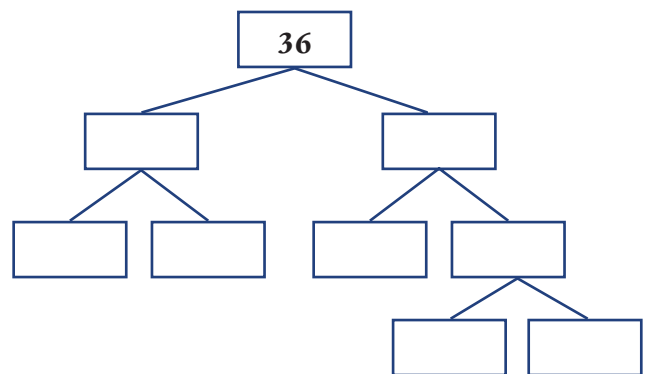












Week 4, worksheet

Adding mixed numbers

An athlete has a training schedule that requires training for $2\frac{3}{4}$ hours on one day and $3\frac{3}{4}$ hours on the next day. For how many hours does the athlete train in two days? Develop a strategy to add $2\frac{3}{4} + 3\frac{3}{4}$. Discuss your strategy with your partner and write it out below.

In another classroom, four teams of students did this problem and came up with different-looking answers.

Team one said that the answer was $5\frac{6}{4}$.

Team two said that the answer was $26/4$.

Team three said that the answer was $6\frac{2}{4}$.

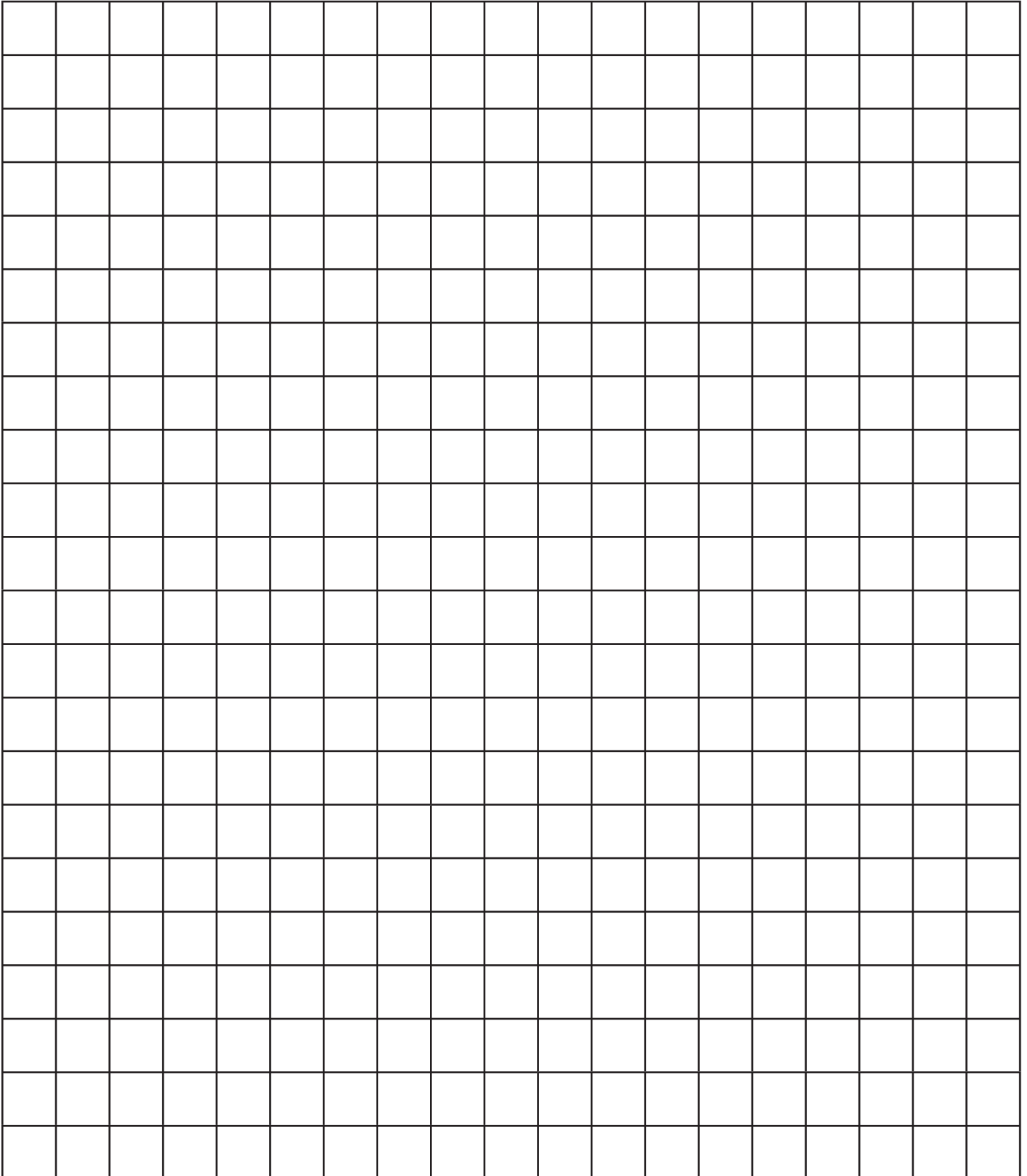
Team four said that the answer was $6\frac{1}{2}$.

Each team says that they have the correct answer. The teacher says that they are all correct. At lunch, the teams are still confused and want you to explain what the teacher means. Working with a partner, prepare an explanation that will convince them that even though their answers look different, they are all correct.

Be ready to share your explanation!

Week 4, handout

Centimetre grid paper



Week 5, worksheet

Creating story problems for multiplication and division of fractions



This is an exercise in 'working backward' from the correct answer.

- 1) On six pages in your notebook (one for each of the following six examples), use a traditional algorithm to find the correct answer. (You can work with your group members.)
 - $2 \times 1/3 =$
 - $3/4 \times 1/2 =$
 - $3/4 \times 2/3 =$
 - $2 \div 1/4 =$
 - $3/4 \div 1/2 =$
 - $2/3 \div 1/6 =$

- 2) Now, in your group, 'work backward' from your correct answer to:
 - write a real-life problem that fits the numbers and operations. (Write a problem that creates a context that helps children understand the mathematics.)
 - draw a model of the problem to visualize the problem.

- 3) What did you discover about:
 - the 'size' of the answer when you multiply a whole number by a fraction?
 - the 'size' of the answer when you multiply a fraction by another fraction?
 - Do you notice any patterns?
 - Can you create (and explain) an algorithm for multiplying with fractions?

 - the 'size' of the answer when you divide a whole number by a fraction?
 - the 'size' of the answer when you divide a fraction by another fraction?
 - Do you notice any patterns?
 - Can you create (and explain) an algorithm for dividing by fractions?

 - the difference between the answers for:
 - $3/4 \times 1/2 =$
 - $3/4 \div 1/2 =$
 - Do you see anything interesting?
 - How does this relate to what you noticed above?

Week 5, worksheet

Four mathematical habits of mind (MHoM)



Look through the maths activities you've done this week. Where do you see the following four mathematical habits of mind? Write your responses individually, then share them with your group.

1) Visualizing

2) Modelling

3) Describing

4) Looking for patterns

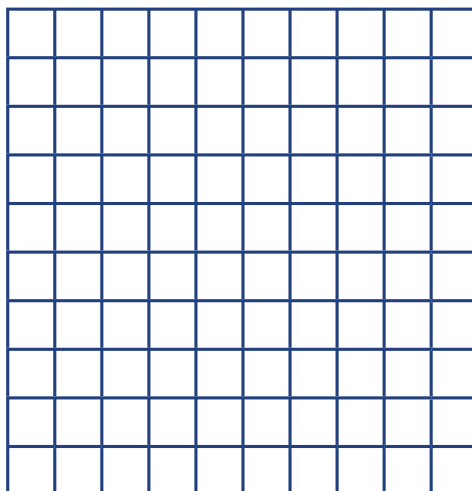
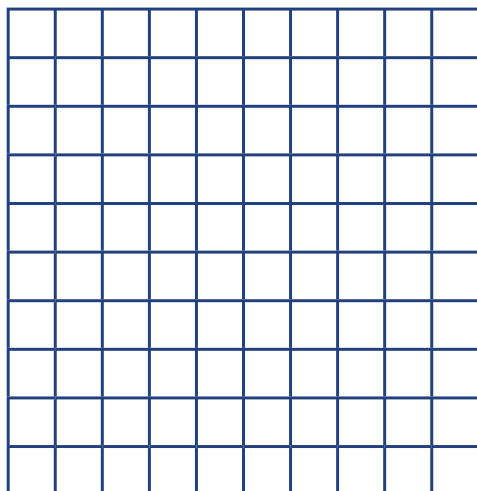
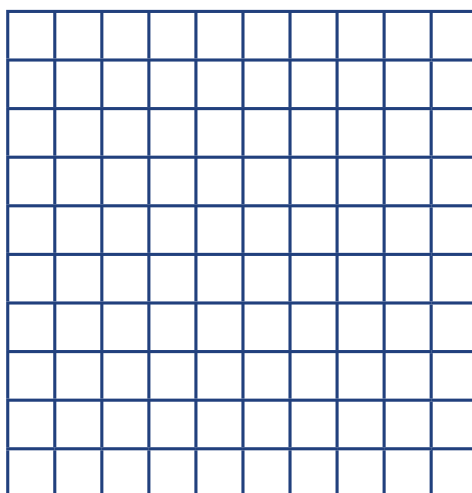
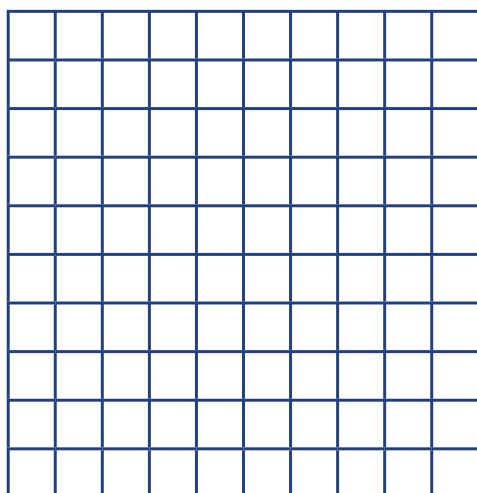
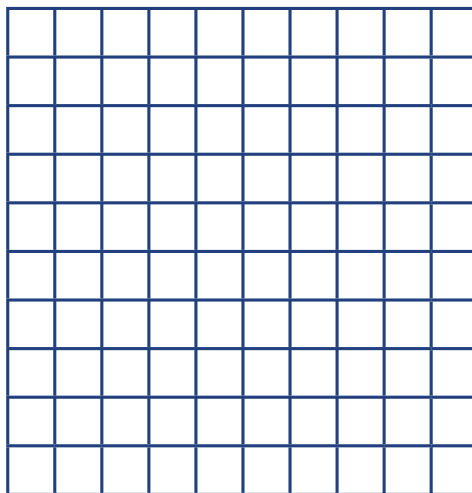
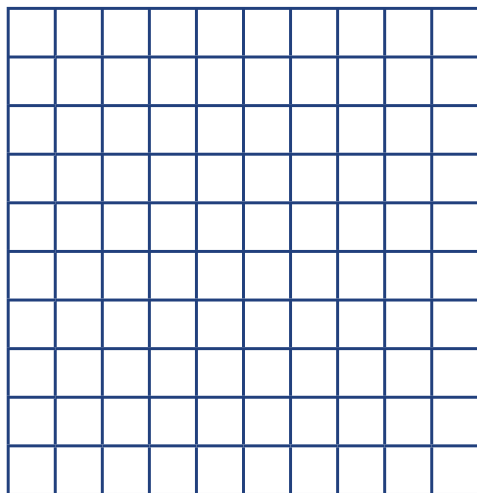
How will you build these four MHoM into your work when you begin teaching?

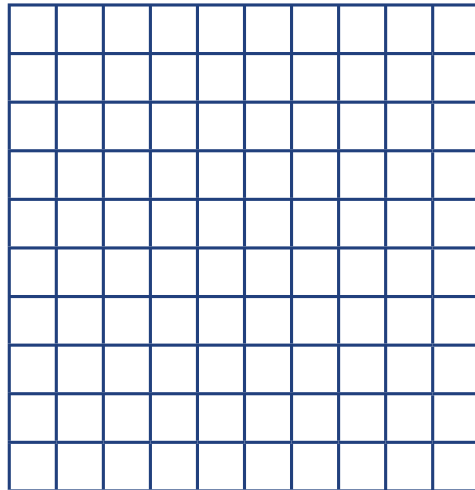
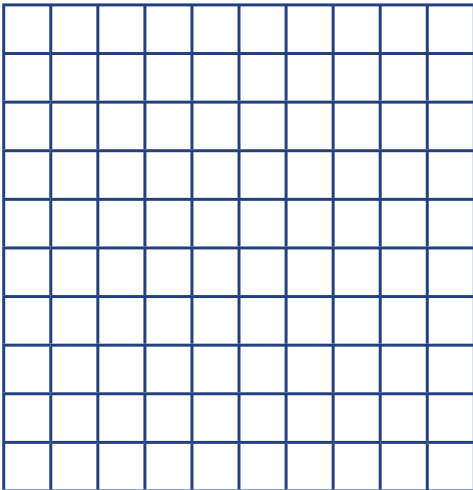
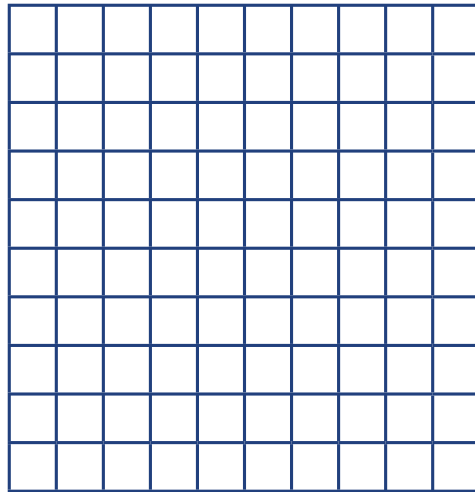
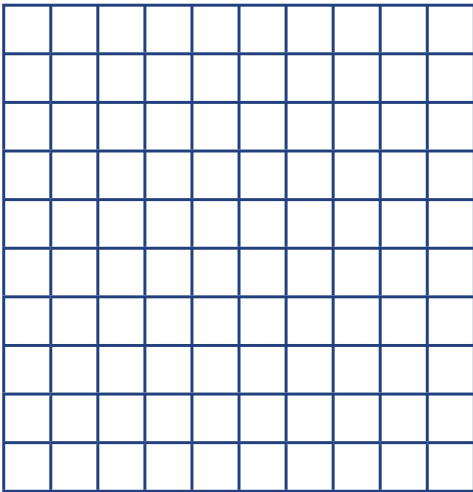
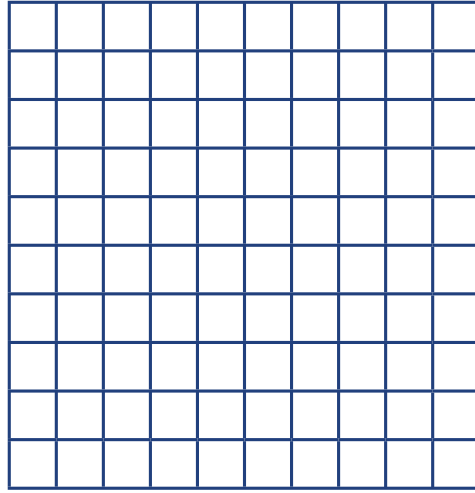
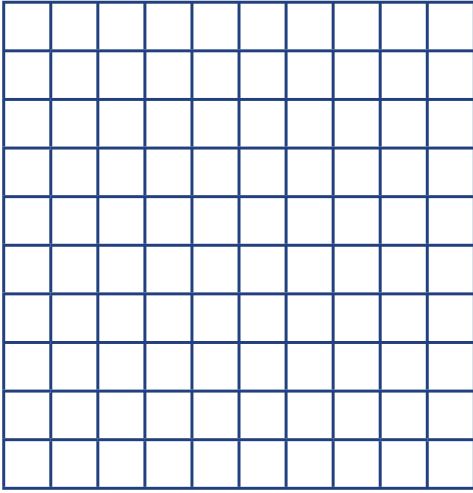
Place a tick mark next to any of these other MHoM that you saw in this week's activities.

- a. Experimenting
- b. Conjecturing
- c. Justifying claims and proving conjectures
- d. Guessing
- e. Seeing regularity in repeated calculations
- f. Generalizing from examples
- g. Challenging solutions, even correct ones
- h. Creating and using alternative representations
- i. Organizing information to discover a pattern
- j. Finding and explaining a pattern
- k. Working the steps of a rule or procedure backward from a correct answer
- l. Justifying why a rule works 'for any number'

Week 6, handout

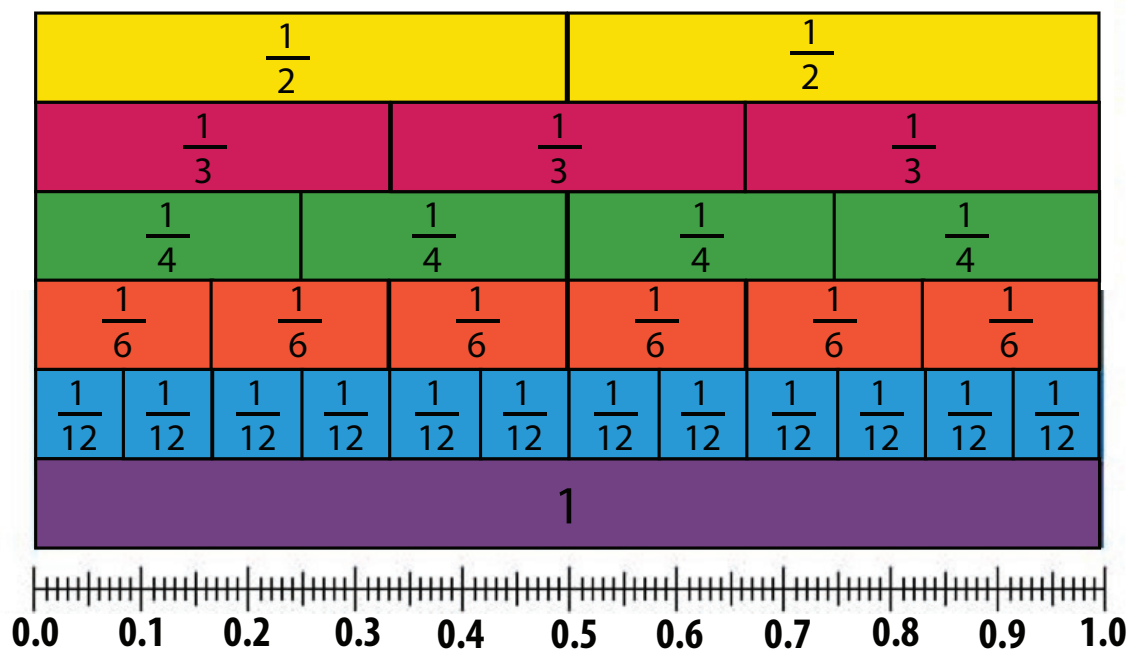
Hundredths grids





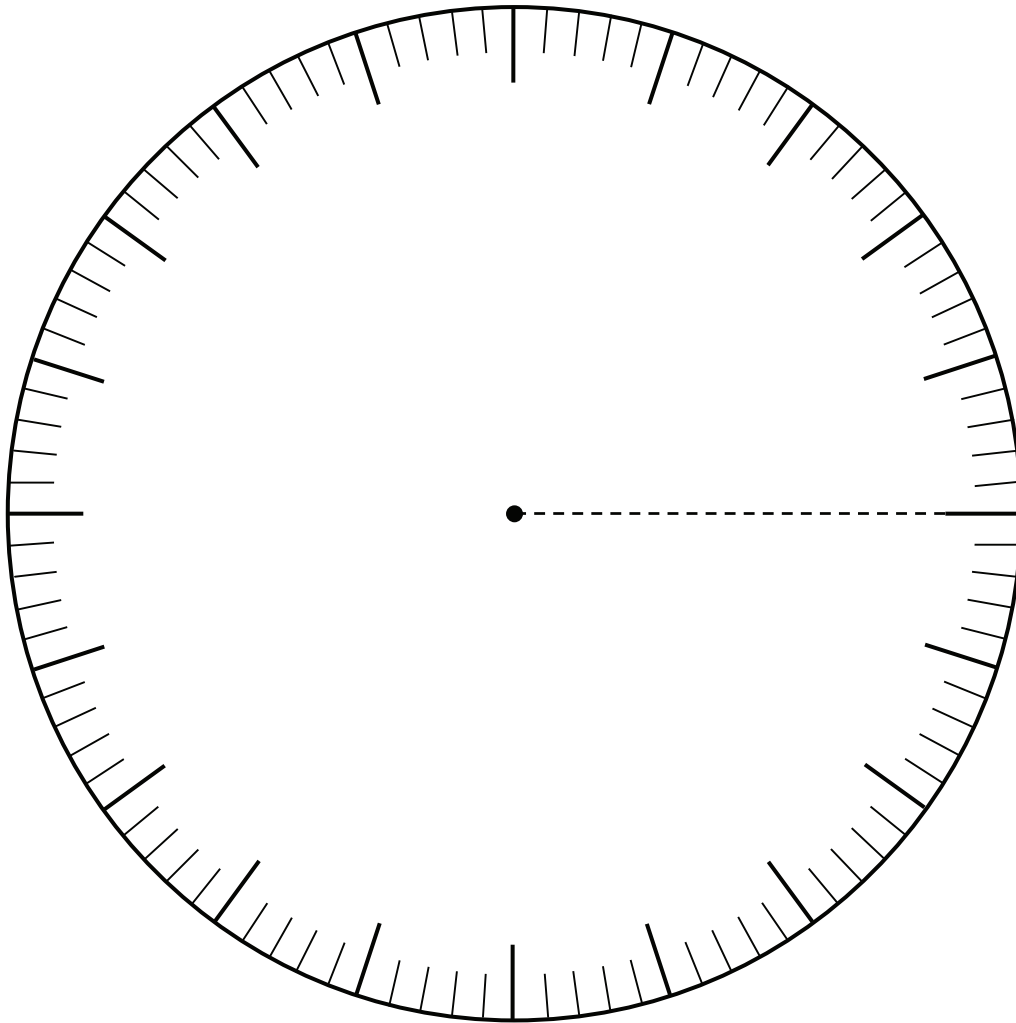
Week 6, handout

Fraction strips



Week 7, handout

Hundredths disk



Week 7, reading

My home library

I love books! The following is a true story.

During 40 years of marriage, I've collected a home library that contains about 400 books.

About 90 of those are maths books and copies of the journal *Teaching Children Mathematics*.

I have 30 cookbooks.

I have 25 books about Italian language study and Italian literature.

I have about 20 classic novels, 13 books of poetry, another 12 books about film, and 30 copies of Shakespeare's plays and commentaries about them.

Being a parent, I collected many, many children's books from the time my sons were babies until they graduated from high school. I estimate that I have at least 180 of their books because I have six boxes of those books stored away.

Think about the information and data above:

- What does my book collection tell you about me?
- About how many books per year have I bought?
- Do you think there were some years when I bought more or fewer than that number of books? Why?
- There are 8 different types of books, which averages to 50 per category. But the data do not indicate that. Why is there a conflict between the average (mean) and the reality of the data?
- On average, how many children's books might be in one box?
- Is estimating a total number of children's books from a sample in one box a valid way of calculating the collection of children's books as a whole?
- Why might I be keeping all those children's books? (Hint: both my sons are married.)

The above narrative gives you discrete, countable data, which means you could create the following data displays. Which do you think would be most useful in communicating the data?

- A bar graph (What scale would you use?)
- A line plot (What would your range on the number line segment be?)
- A pictograph using book icons (What would you use as your key?) Is it necessary on a pictograph to differentiate between 13 poetry books and 12 books about film in a collection of 400 books?
- A circle graph showing how 8 different types of books are distributed over a 400-book collection.

Create that circle graph now.

Think about:

- What do you need to do to translate the above data into a format that can be used to construct your circle graph?
- What did you learn in the units on numbers and operations and geometry that can help you complete this task?
- How you would help youngsters create a circle graph (if the data were simpler than this)?

Week 7, worksheet



Lesson planning: The 50-minute bean chart

In planning a lesson, you need to consider not only how much time you have for the lesson itself, but also for time to:

- Get students settled
- Check homework
- Transition from one part of the lesson to another
- Distribute materials
- Deal with discipline
- Assign (and explain) homework
- And so forth

Using the launch-explore-summarize model of lesson design, the following chart may help you see how precious your time is! You have 50 beans, which represent a 50-minute lesson. How will you divide your time? Do this activity individually first, then share with group members how you came to your decision for pacing your lesson.

Activity	Beans	Time
Launch		
Explore		
Summarize		
Everything else		

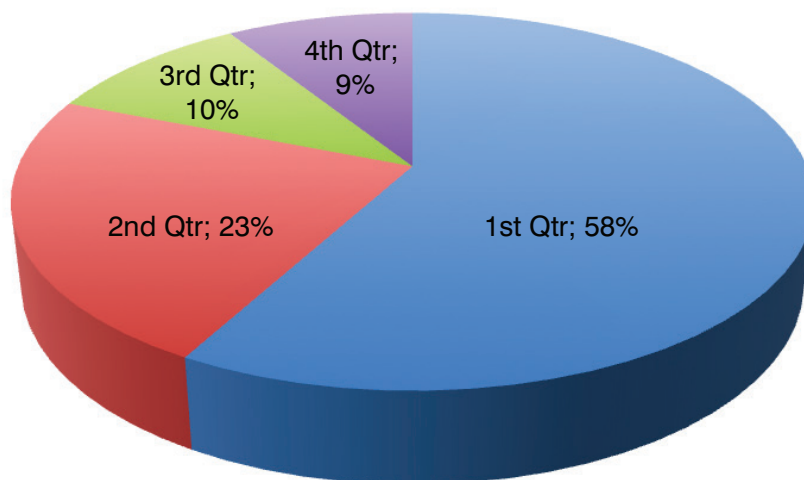
Week 7, handout

Analysing pie charts



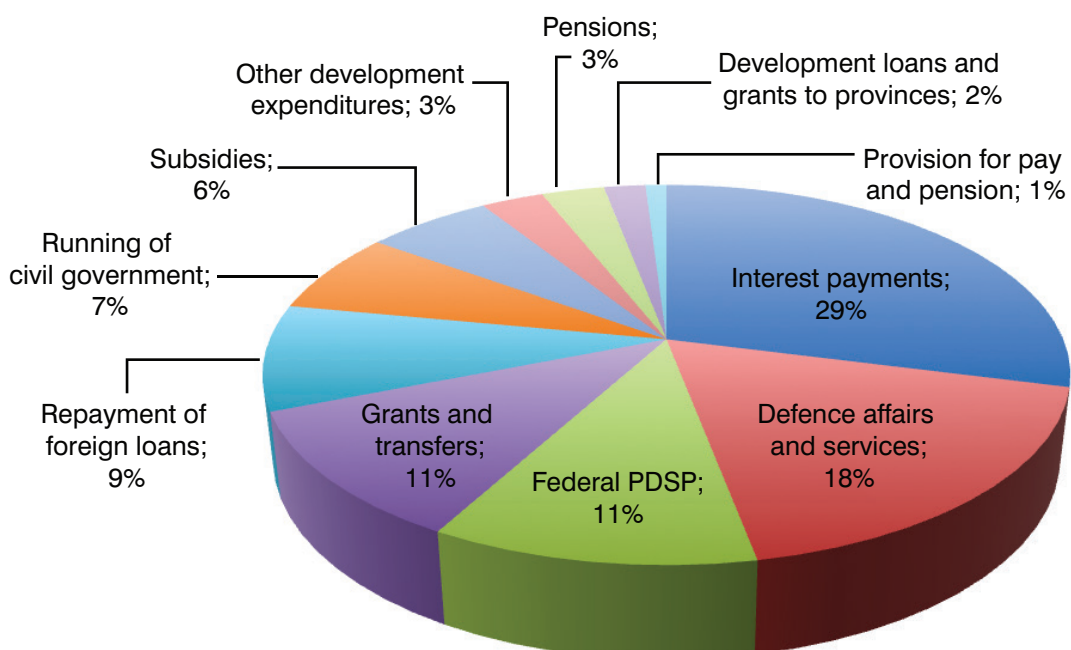
Example 1

Sales of ice cream by quarter



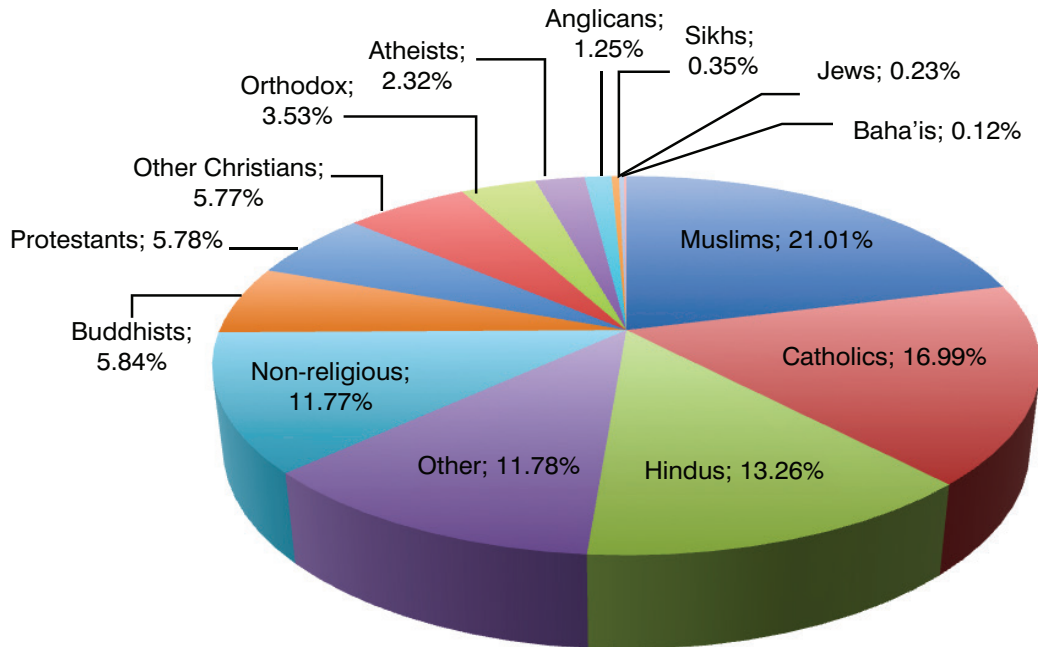
Example 2

Budget 2011-2012: Expenditures in Rs billion



Example 3

World religions by percentage (2007 est.)



Week 8, worksheet

Exploring similarity: Triangles and rectangles



- 1) On lined paper, with one of the lines as a base, draw a large triangle '10 lines' long. Label this triangle's vertices A, B, and C.
- 2) Draw a line inside triangle ABC, parallel to its base line.
- 3) Label the vertices of this new triangle $A' B' C'$. (These new vertices are called 'A-prime, B-prime, C-prime'.)
- 4) Note that the apex angle of both triangle ABC and triangle $A' B' C'$ are the same.
- 5) Use a protractor to measure the base angles of both triangle ABC and triangle $A' B' C'$.
- 6) What do you notice?
- 7) Using a ruler, measure the side lengths of the larger triangle (ABC). Then measure the corresponding side lengths of the smaller triangle ($A' B' C'$). How do the side lengths compare?
- 8) What do you notice?
- 9) Compare your results to your classmates' results.
- 10) Even if your original triangle had a different shape than the triangles of your classmates, was anything consistent?
- 11) Do you think this activity would have the same results if you drew a large rectangle and followed the above directions?
- 12) Try it! Compare the corresponding side lengths and corresponding angles of your rectangles. What do you notice?

Week 8, worksheet

Using a grid to enlarge or shrink a shape

Create a new figure that is mathematically similar to Detective Duck. Draw it next to him on the same grid.

Enlarge Detective Duck by any scale factor you choose.

What did you find difficult?

Does your new figure look mathematically similar? Or does it look distorted? (That is OK. Think about where you 'went wrong'.)

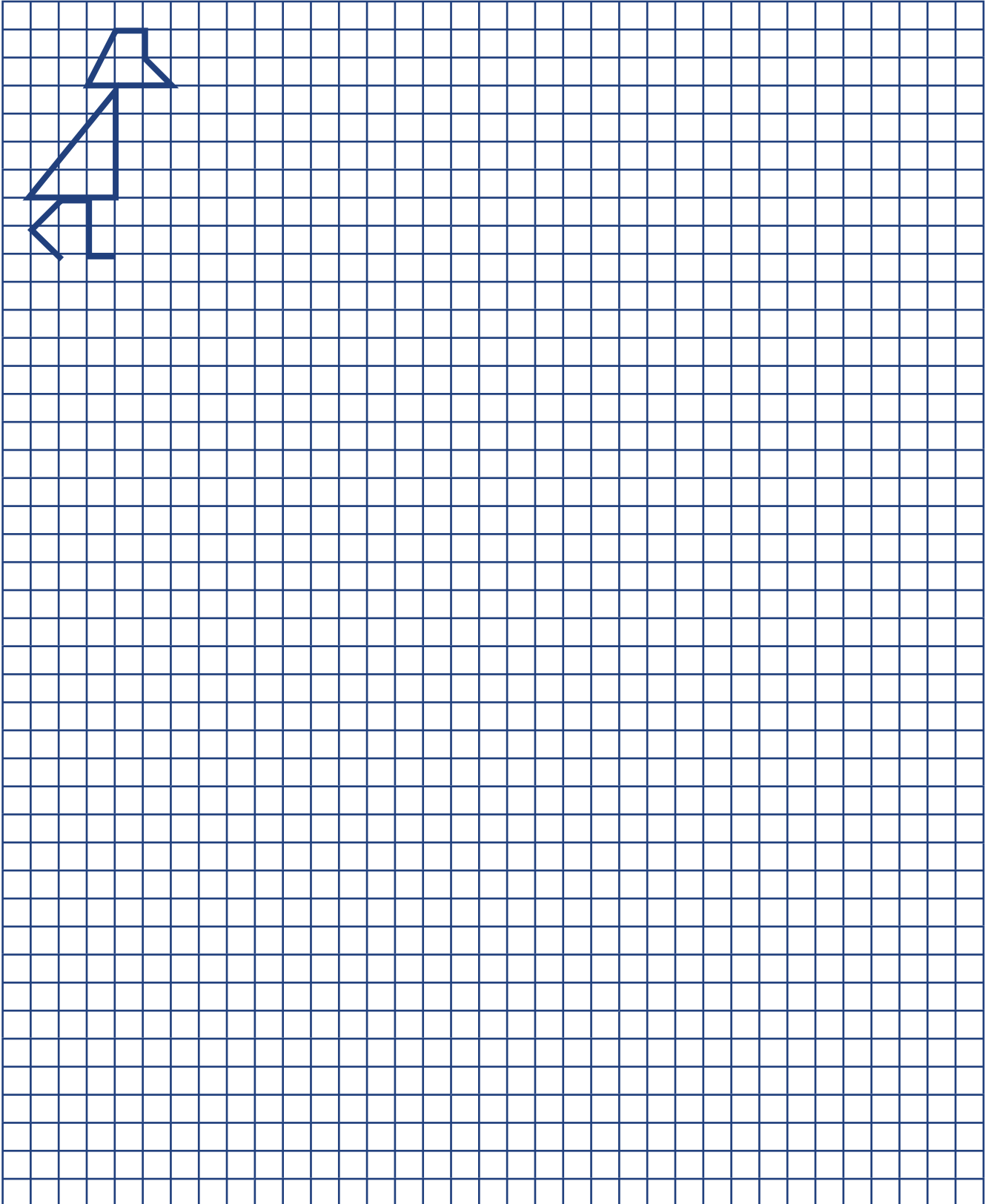
Given what you know about scale factor and area, what generalization can you make about the size of your new image?

What would it have been like if this activity had been introduced on Session 1 this week?

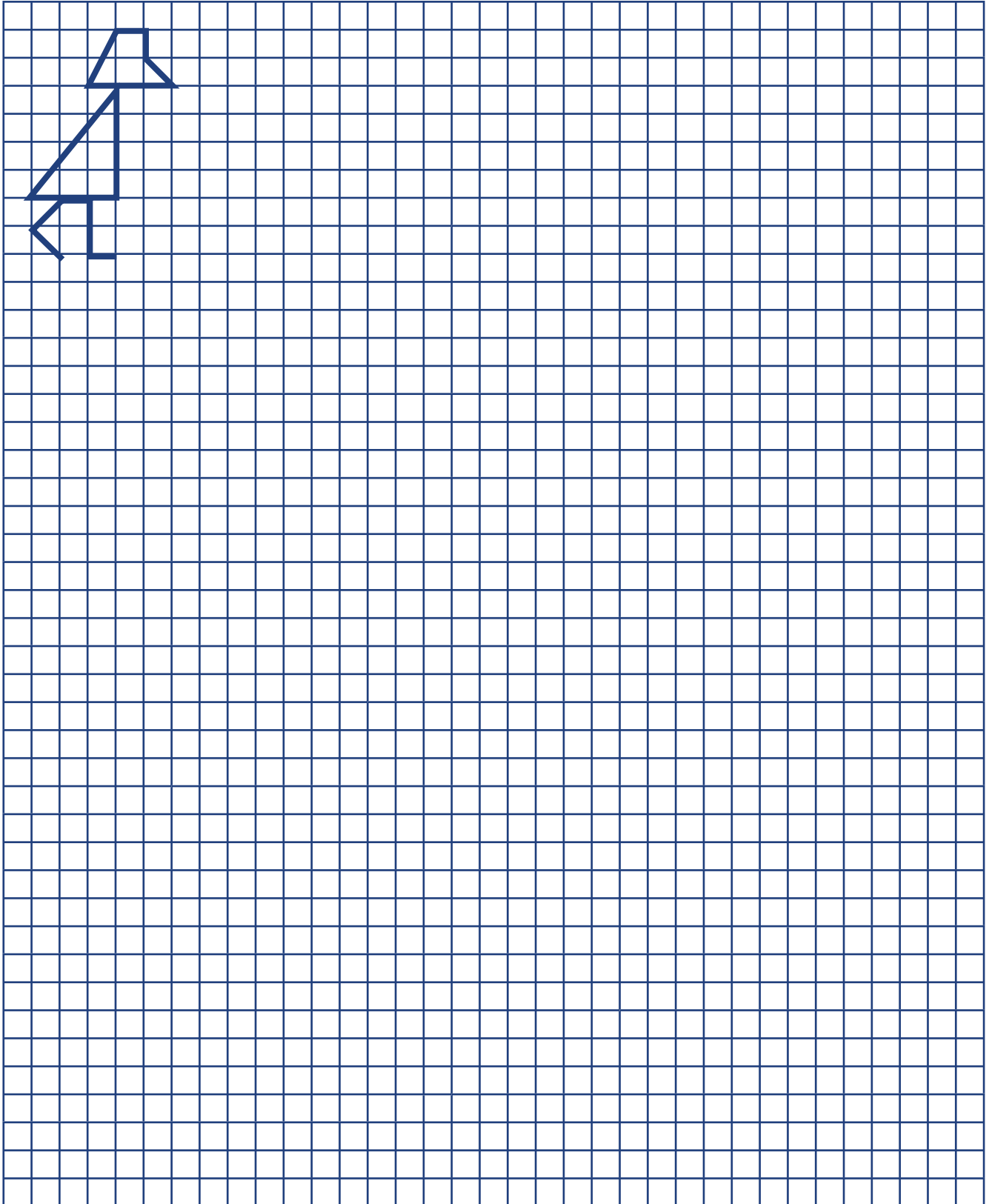
How did scaffolding throughout the week help you when completing this assignment?

Week 8, handout

Detective duck



Detective duck



Week 9, worksheet

Proportion word problems



- 1) How many spoonfuls of sugar do you think you would need to make a litre of *nimbu pani*? How many spoonfuls of sugar would you need to make four litres of *nimbu pani*?

- 2) Research suggests that for every left-handed person, there are about nine right-handed people. Suppose you have 30 children in your class when you become a teacher. Write and solve a proportion predicting the number of children who are right-handed. Bonus question: Does this proportion describe the 'handedness' in your college or university class?

- 3) Remember your work with geometric proportions. When you use a copier, you can enlarge or reduce an original. You have an original picture of a tiger that has a length of 20 centimetres. You reduced it on a copy machine so that its new length is 15 centimetres. If the width of the original was 16 centimetres, what is the width of the reduced copy? Bonus question: What was the scale factor?

- 4) The ratio of an object's weight on Earth to its weight on the Moon is 6:1. The first person to walk on the Moon was Neil Armstrong. He weighed 75 kg on Earth. How much did he weigh on the Moon? Bonus question: What do you weigh? What would you weigh on the Moon?

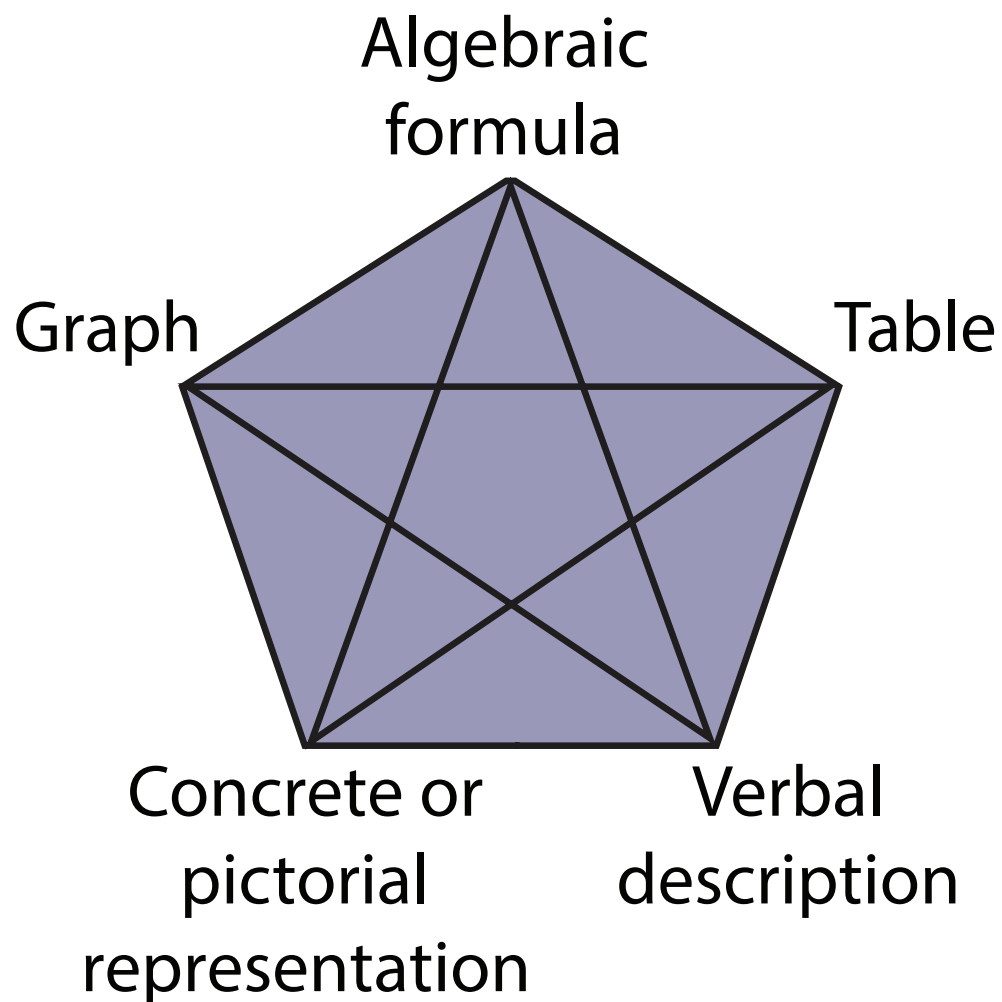
Adapted from 'Proportion Word Problems', at:

- http://eclass1.wsd.k12.ca.us/moodle/pluginfile.php/3581/mod_resource/content/1/Proportions_Word_Problems.pdf

Week 10, handout



Algebra star



Week 10, worksheet and transparency

Milk shakes



You and your friend Tahira both love milk shakes.

A shop in your neighbourhood has a Frequent Customer plan:

- 1) Buy a 'Frequent Customer' card.
- 2) Then pay a reduced price for each milk shake whenever you show your card.

By the end of the first month, you bought 8 milk shakes, paying 1650 rupees, including the fee for your customer card.

Tahira bought 6 milk shakes, paying 1400 rupees, including her customer card.

- 1) What did the customer card cost?

- 2) What did each milk shake cost?

- 3) How can you solve these problems?

Week 10, worksheet

Simultaneous equations

Renting a car

Nadya wants to rent a car for one week. She calls two car rental companies to find out prices.

Ahmed's Rentals rents a car for 10,000 rupees for one week, with 100 free kilometres, and 11 rupees per kilometre over 100 kilometres.

Saleem's Rentals has the same car for 7500 rupees for one week, with 150 free kilometres, and 15 rupees per kilometre over 150 kilometres.

Nadya's trip is 432 kilometres.

- 1) Which company has the better deal?
- 2) How many kilometres would Nadya have to drive before she would spend the same amount at either company?

Be careful! Nadya has to return from her trip, too.

Mobile phone plans

You are trying to decide on a mobile phone plan.

One company has a base rate of 4000 rupees per month. It includes a free phone, and you receive 300 free minutes per month. It costs 12 rupees per minute for each minute over the 300 free minutes.

Another company has a base rate of 3600 rupees per month. You must buy the phone for a one-time cost of 1000 rupees, and you receive 350 free minutes per month. It costs 8 rupees per minute for each minute over the free 350 minutes.

- 1) If you talk about 400 minutes per month, which company will be a better choice for you?
- 2) How many minutes could you talk on each plan before spending the same amount of money?

Hints for solving

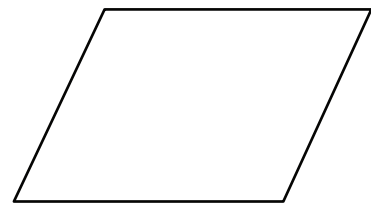
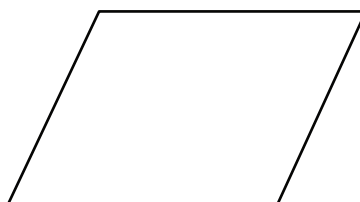
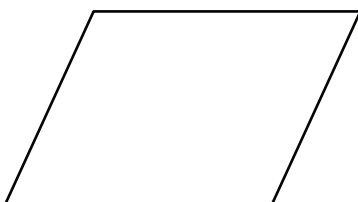
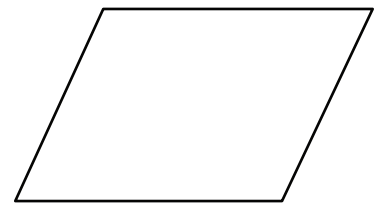
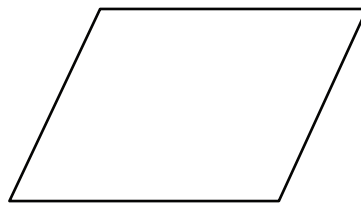
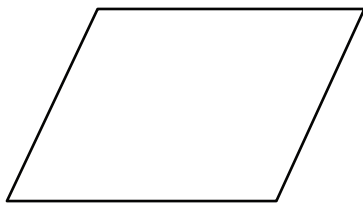
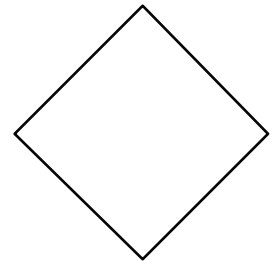
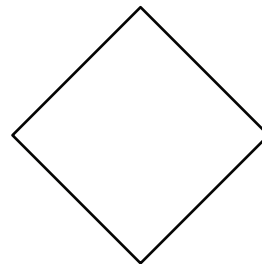
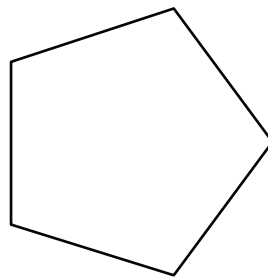
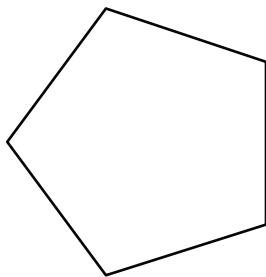
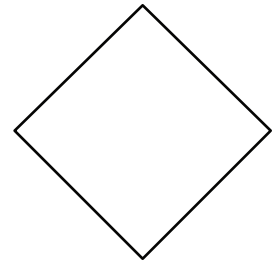
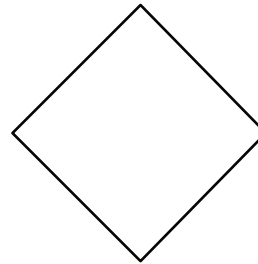
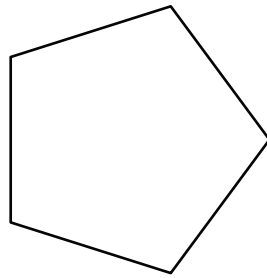
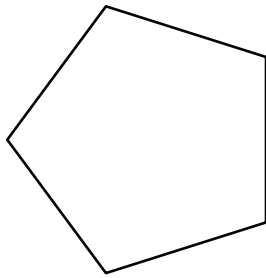
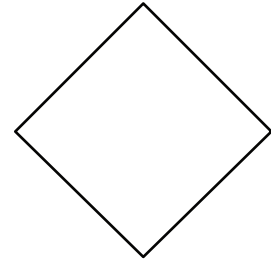
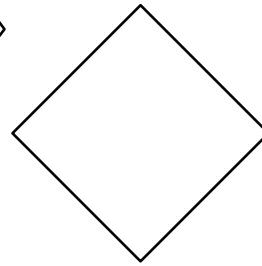
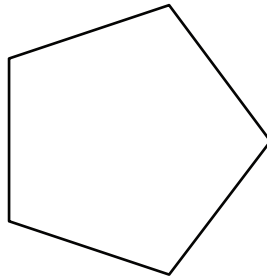
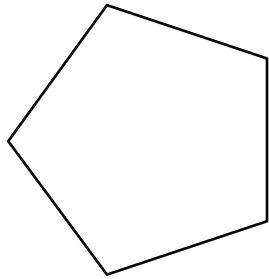
What are the unknowns? Create a variable for each unknown.

Try using multiple representations:

- Create a table of values.
- Create a linear equation for each situation using the variables you chose.
- Graph your two situations on the same sheet of graph paper. What does the intersection of the two lines tell you?

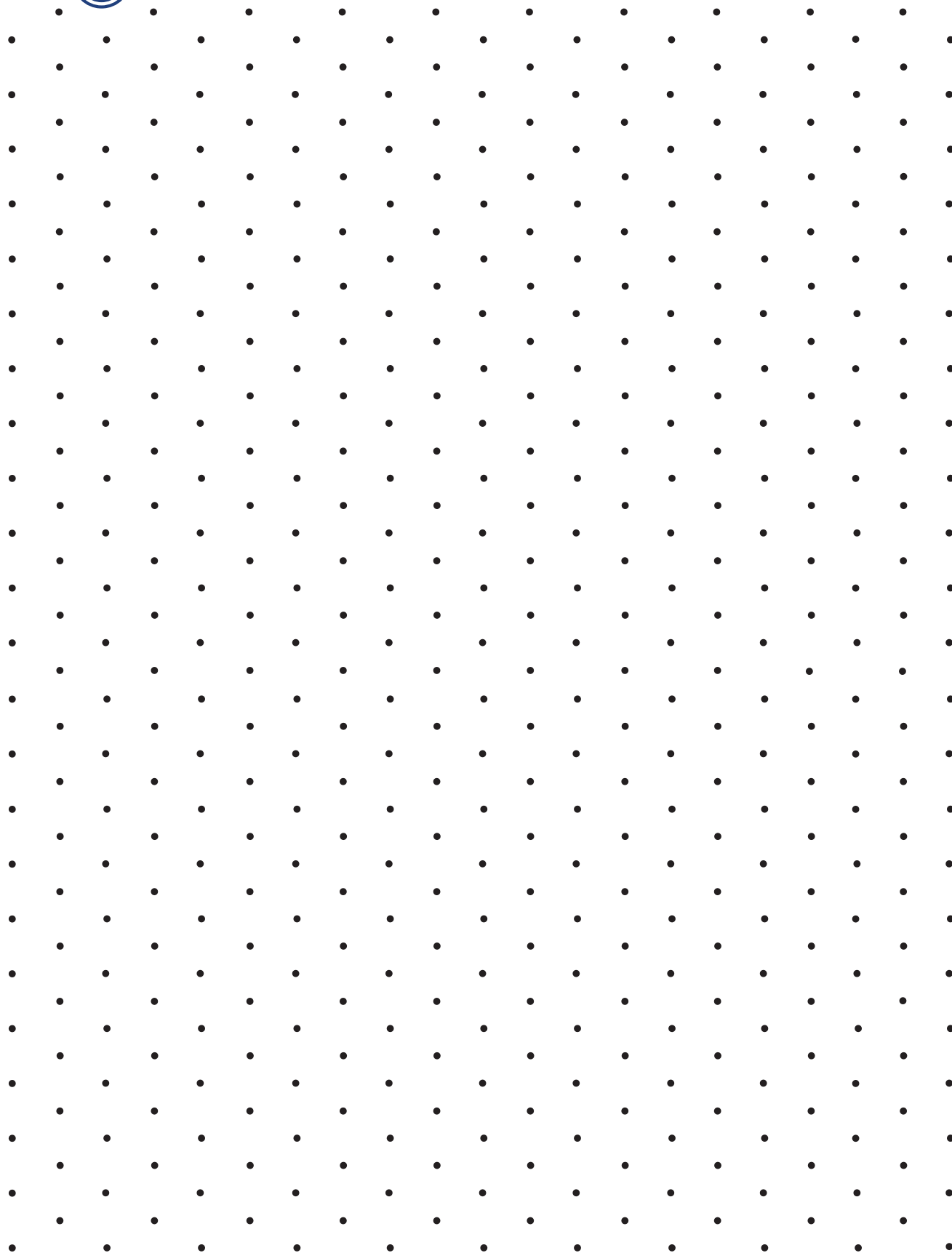
Week 11, handout

Rotation cut-outs



Week 12, handout

Isometric dot paper



Week 13, worksheet

Measurement task cards

S

You're proud of your collection of cricket trophies. You're going to build a display shelf. What are the dimensions of the shelf? Why?

Your friend needs a piece of ribbon that is 12 inches long. You have a whole roll of ribbon. How will you know how much to give her?

You need a mat that is 40 cm x 60 cm. How will you know that you have selected a mat that is the correct size?

You are wrapping a gift that is 25 cm long, 14 cm wide, and 6 cm tall. Will one metre of paper that is 50 cm wide be enough? How do you know?

You want to buy a storage unit for your DVDs. The unit is 1 metre 20 cm tall and has 4 movable shelves. How will you place the shelves? Why?

You are shopping to buy a new wallet. How will you be sure the wallet will fit in your pocket (even when it is full)?

You are helping set up the area for the school sports day. You are to place hoops about 40 cm apart. How can you make sure the distance is correct?

You want to buy the biggest possible bottle of shampoo. Your shelf has 25 cm of space above it. How can you make sure the shampoo bottle will fit?

You need new shoelaces. Will laces that are 30 cm long be long enough? How do you know?

Week 14, handout

Close to 100

You are going to play a game with a partner that involves the addition of two-digit numbers. (Remember, the whole number six can be written in tens-and-ones format as 06.)

This is a game often used with primary grade students. The object of the game is to add two two-digit numbers whose sum is close to 100. There are several parts to this game.

- 1) Create two two-digit numbers from a set of four randomly selected one-digit numbers.

For example, if your numbers are 3, 7, 9, and 0, what two-digit numbers can you make?

Write all those numbers in your notebook.

Consider:

- What mental skills were involved in this first part of the game?
 - How did you decide to combine the digits?
 - If there was a pattern in your thinking, what was it?
 - Could you have done this task as mental arithmetic, perhaps thinking of the numbers in least to greatest order, without writing all the options down?
 - How do you think a primary-grade student would approach this part of the game?
 - Does this part of the game involve higher- or lower-level thinking?
- 2) Determine which two, of all possible combinations, could be added to create a sum 'close to 100'. (This sum could be either less than or greater than 100, but it should be close.)
 - What mental skills were involved in this second part of the game?
 - How did you choose your two-digit numbers?
 - If there was a pattern in your thinking, what was it?
 - How do you think a primary-grade student would approach this part of the game?
 - Does this part of the game involve higher- or lower-level thinking?

- 3) Calculate the sum of the two numbers you have chosen. Compare your sum to 100. Determine the difference between 100 and your sum. Write this down.

Does this part of the game involve higher- or lower-level thinking?

- 4) The winner of the game is the player whose 'sum of the differences' is 'closest to 100'.

Consider:

- What were the levels of lower- versus higher-level thinking for the different parts of the game?
- At what point might a calculator rather than paper and pencil have advanced students' learning while keeping them engaged in the game?

Week 14, handout

Instructional technology sampler

‘The evaluation of materials for mathematics teaching should be an essential aspect of programme planning’.

Look through these websites and evaluate them. What criteria will you use to determine their mathematical worth and usefulness?

Numbers and operations: Integer addition and subtraction

- Number-line model:
 - <http://tinyurl.com/Integers-Interactive>
- Zero-sum pairs model:
 - Addition:
 - <http://tinyurl.com/Integer-Chip-Applet>
 - Subtraction:
 - <http://tinyurl.com/Integer-Subtr> (Click OK for Java)

Algebra: Balancing equations

- Numbers:
 - <http://tinyurl.com/Balance-Eqat-Numeric>
- Algebraic:
 - <http://tinyurl.com/Balance-Eqat-Algebr>

Geometry:

- Surface area:
 - <http://tinyurl.com/Surface-Area-vs-Volume>
- Pattern blocks:
 - <http://tinyurl.com/Virt-Pat-Blocks>
- Interactive geoboard:
 - <http://tinyurl.com/Virt-Geoboard>

Graphing calculator

This free online graphing calculator requires no subscription, no downloading, no software, and has no advertising.

In line with the idea of multiple representations, it also creates a table of values next to the graph. Students see the equation, the graph, and the table all on the same page.

- <http://tinyurl.com/Free-Graph-Calc>

Week 14, worksheet

Methods of alternative assessment



Methods of alternative assessment should make higher cognitive demands while at the same time responding to different learning needs.

Perhaps a teacher's most important alternative assessment tools are diagnostic:

- Your observational notes during the exploration
- Your analysis notes about the summary
- Error pattern analysis of student work
- Rubrics you create to clarify standards of quality work

Look through this list of other alternative ways to differentiate assessment. Put a tick mark next to those you remember being used during this course.

- Group projects
- Journals
- Games
- Puzzles
- Oral presentations
- PowerPoint presentations
- Posters
- Drawings and diagrams
- Music and rhythm
- Physical and kinesthetic activities
- Exhibitions
- Portfolios of best work
- Portfolios showing progress
- Collections of relevant websites
- Statistical reports
- Research reports
- Observational reports
- Working with manipulatives
- Constructions
- Multiple representations
- Writing word problems
- Writing and putting on a play

Appendix



Professional standards for teaching mathematics

In 2009, the Ministry of Education passed into policy a set of National Professional Standards for Teachers in Pakistan (NPSTP). The 10 standards describe what a teacher should know and be able to do.

The following is a list of standards specific to the teaching of mathematics. They were developed to be used in conjunction with the three mathematics courses in the B.Ed. (Hons) Elementary/ADE programme. They provide a description of the knowledge, skills, and dispositions a teacher requires to teach mathematics.

This set of standards for teaching maths is linked to the NPSTP. The first standard in the NPSTP concerns Subject Matter Knowledge – the knowledge, skills, and dispositions a teacher requires to teach the National Curriculum. In the NPSTP, knowledge, skills, and dispositions are described in general terms for all subjects. Here, they are described specifically for teaching mathematics.

The standards for teaching mathematics may be used by Instructors and Student Teachers in a variety of ways, such as for assessment (including self-assessment) and planning instruction. The standards may also be used as part of instruction. Helping Student Teachers deconstruct and understand the standards (and how they apply in the classroom) will help them learn about teaching mathematics.

Subject Matter Knowledge (Teaching Mathematics)

Knowledge and understanding

Teachers know and understand the following:

- the National Curriculum framework for mathematics
- the basic concepts and theories of maths; the history and nature of mathematics and how to explore mathematical relationships, to represent data visually and symbolically, and to make generalizations that predict future outcomes; and the structure and process of acquiring additional knowledge and skills in mathematics
- in-depth knowledge of mathematics and its relationship to other content areas
- the relationship between mathematics and other disciplines
- the usability of mathematics in everyday life
- that mathematics is not a discrete set of facts, but a relationship between a wide range of concepts across the five mathematical strands
- that maths content and pedagogy cannot be separated and that an effective lesson plan makes use of content and pedagogy simultaneously
- the evolving nature of mathematics and its content
- that learning mathematics is an ongoing and reflective practice.

Dispositions

Teachers value and are committed to doing the following:

- facilitating the construction of knowledge in multiple ways
- deepening learners' understanding and skills
- making mathematics integral for application to real-world situations
- recognizing and building on the diverse talents of all students and helping them develop self-confidence and mathematical literacy
- believing that *all* students can make meaning in mathematics through their own constructions and be successful users of mathematical knowledge and skills
- acknowledging the intellectual learning and growth of all students (boys and girls)
- creating an engaging learning environment that promotes knowledge building through student interaction (discussion, asking questions, and sharing ideas and examples) with peers and teachers
- providing meaningful learning experiences that deepen math content knowledge (rather than emphasizing rote memorization of isolated facts)
- teaching students to think critically about information and how to make informed decisions that affect their lives.

Performance and skills

Teachers demonstrate their knowledge and understanding by doing the following:

- selecting mathematical content from the National Curriculum that reflects and extends students' prior mathematical knowledge and understanding
- using tools and methodologies for inquiry appropriate to mathematics and the content being taught
- effectively explaining mathematical content from multiple perspectives and in multiple ways that are appropriate for the students they are teaching
- giving examples of how mathematical content applies to daily life
- asking engaging questions that stimulate thinking and reasoning rather than the memorization of facts
- using of math manipulatives to deepen students' understanding of mathematical concepts
- modelling ways to help students construct their knowledge of mathematics
- analysing materials (e.g. textbooks) to see if they are useful and appropriate resources for teaching and learning mathematical concepts and skills detailed in the National Curriculum
- arousing curiosity about mathematical ideas
- organizing opportunities for students to discuss and develop their understanding of mathematical concepts.

Adapted from National Professional Standards for Teachers in Pakistan. Policy and Planning Wing, Ministry of Education, Government of Pakistan (2009). Available from:

➤ <http://unesco.org.pk/education/teachereducation/files/National%20Professional%20Standards%20for%20Teachers.pdf>

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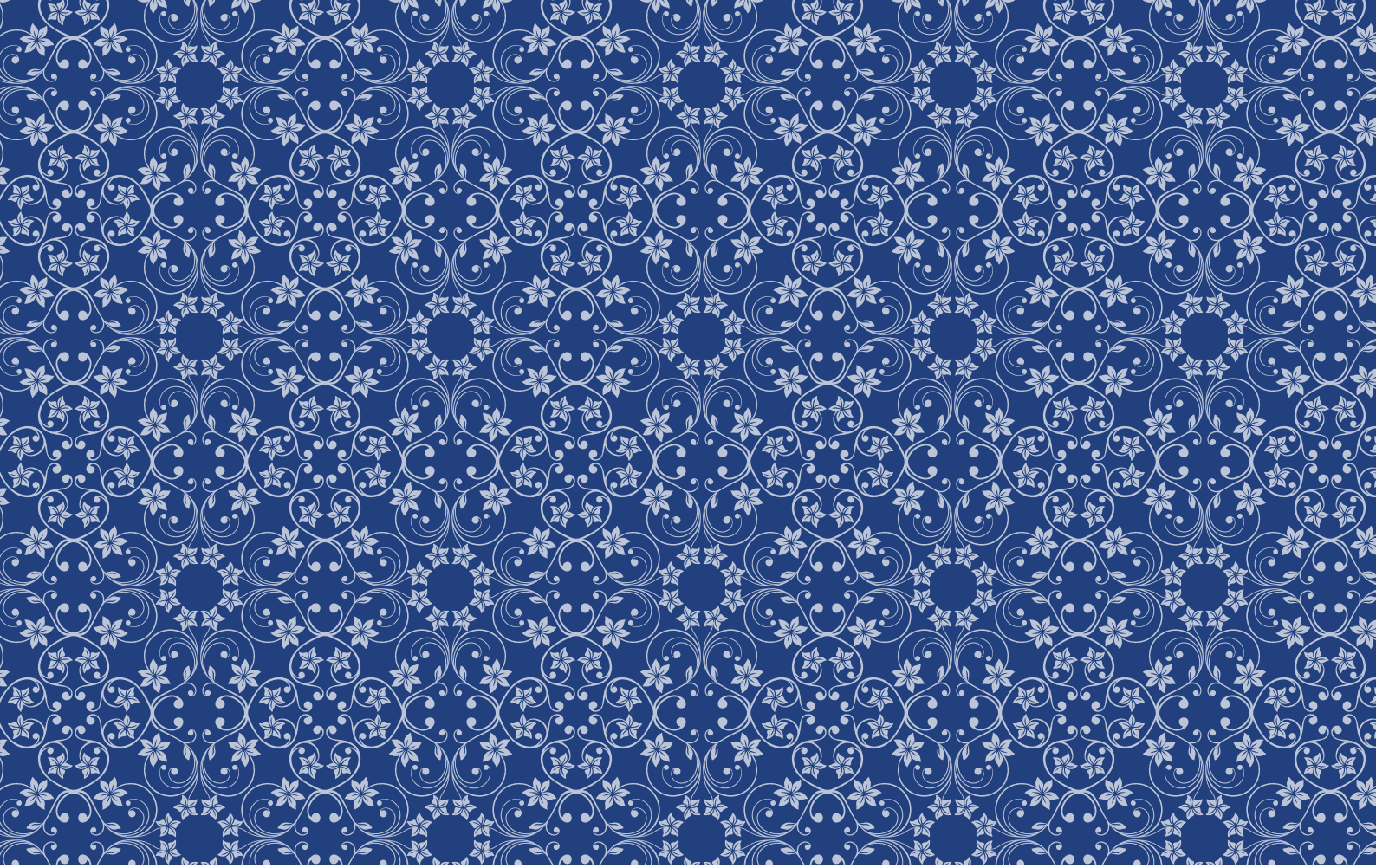
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