semester 2

Mathematics I: General Mathematics

COURSE GUIDE

Associate Degree in Education/
B.Ed. (Hons) Elementary

2012

Higher Education Commission

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Higher Education Commission

Foreword

Teacher education in Pakistan is leaping into the future. This updated Scheme of Studies is the latest milestone in a journey that began in earnest in 2006 with the development of a National Curriculum, which was later augmented by the 2008 National Professional Standards for Teachers in Pakistan and the 2010 Curriculum of Education Scheme of Studies. With these foundations in place, the Higher Education Commission (HEC) and the USAID Teacher Education Project engaged faculty across the nation to develop detailed syllabi and course guides for the four-year B.Ed. (Hons) Elementary and the two-year Associate Degree in Education (ADE).

The syllabi and course guides have been reviewed by the National Curriculum Review Committee (NCRC) and the syllabi are approved as the updated Scheme of Studies for the ADE and B.Ed. (Hons) Elementary programmes.

As an educator, I am especially inspired by the creativity and engagement of this updated Scheme of Studies. It offers the potential for a seismic change in how we educate our teachers and ultimately our country's youngsters. Colleges and universities that use programmes like these provide their students with the universally valuable tools of critical thinking, hands-on learning, and collaborative study.

I am grateful to all who have contributed to this exciting process; in particular the faculty and staff from universities, colleges, and provincial institutions who gave freely of their time and expertise for the purpose of preparing teachers with the knowledge, skills, and dispositions required for nurturing students in elementary grades. Their contributions to improving the quality of basic education in Pakistan are incalculable. I would also like to thank the distinguished NCRC members, who helped further enrich the curricula by their recommendations. The generous support received from the United States Agency for International Development (USAID) enabled HEC to draw on technical assistance and subject-matter expertise of the scholars at Education Development Center, Inc., and Teachers College, Columbia University. Together, this partnership has produced a vitally important resource for Pakistan.

PROF. DR SOHAIL NAQVI
Executive Director
Higher Education Commission
Islamabad

How this course guide was developed

As part of nation-wide reforms to improve the quality of teacher education, the Higher Education Commission (HEC) with technical assistance from the USAID Teacher Education Project engaged faculty across the nation to develop detailed syllabi and course guides for the four-year B.Ed. (Hons) Elementary and two-year Associate Degree in Education (ADE).

The process of designing the syllabi and course guides began with a curriculum design workshop (one workshop for each subject) with faculty from universities and colleges and officials from provincial teacher education apex institutions. With guidance from national and international subject experts, they reviewed the HEC Scheme of Studies, organized course content across the semester, developed detailed unit descriptions and prepared the course syllabi. Although the course syllabi are designed primarily for Student Teachers, they are useful resources for teacher educators too.

In addition, participants in the workshops developed elements of a course guide. The course guide is designed for faculty teaching the B.Ed. (Hons) Elementary and the ADE. It provides suggestions for how to teach the content of each course and identifies potential resource materials. In designing both the syllabi and the course guides, faculty and subject experts were guided by the National Professional Standards for Teachers in Pakistan 2009 and the National Curriculum 2006. The subject experts for each course completed the initial drafts of syllabi and course guides.

Faculty and Student Teachers started using drafts of syllabi and course guides and they provided their feedback and suggestions for improvement. Final drafts were reviewed and approved by the National Curriculum Review Committee (NCRC).

The following faculty were involved in designing this course guide: Shabana Saeed, GCET (F) Rawalakot; Saima Khan, University of Education, Lahore; Khalid Pervez, GCET Kasur; Dr Shahid Farooq, IER University of the Punjab, Lahore; Muhammad Zaman, BoC Sindh, Jamshoro; Muhammad Rauf, IER University of Peshawar; Noor Alam, GCET (M) Lalamusa; Shereen Taj, University of Balochistan, Quetta; Zakia Ishaq, GCEE (F) Pishin; M. Nadeem, RITE (M) DI Khan; Zohra Khatoon, University of Sindh; Shoukat Usmani, GCET (M) Muzaffarabad; Ijaz-Ur-Rauf, GCET Shahpur Sadar, Muhammad Asim, University of Karachi; Rashid Ahmed Noor, RITE (M) Peshawar; Muhammad Rafique, GCET Mirpur; Farjana Memon, GECE (W) Hyderabad; Abdul Khaliq, BoC, Quetta; Muhammad Wasim Uddin, RITE (M) Haripur; Muhammad Afzal, University of Education, Lahore; Gul Muhammad, GCEE Quetta; Shabana

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Date of NCRC review: 3 March 2012

NCRC Reviewers: Dr Imran Yousuf, Arid Agriculture University, Rawalpindi; and Dr Tayyab, Foundation University, Islamabad.

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About the course guide

This Course Guide includes a syllabus and planning guide. The syllabus is primarily for Student Teachers. The planning guide is intended for faculty teaching the course.

The planning guide provides suggestions, resources, and links to resources for teaching the course.

The planning guide and course is divided into four units. Each unit is further broken down into weeks.

For each week, the planning guide provides faculty notes that introduce and discuss the content for teaching and learning. After the faculty notes, three session plans are provided.

Each session plan is organized into five parts:

- What do Student Teachers need to know?
- How do children think about these concepts?
- What is essential to do with Student Teachers?
- Activities with Student Teachers
- Assignments

Session plans contain links to useful resources for faculty and Student Teachers. Some of the session resources are also reproduced in the planning guide.

The planning guide is a guide only. Faculty should use their professional judgement to decide if and how to use the planning guide and to modify and adapt the content for their context.

The writers of the course aimed to respect copyright and have sought permission to use copyrighted material when necessary. Please contact EDC in case of questions or concerns about any of the materials used, at:

➢ www.edc.org

Syllabus

MATHEMATICS I:
GENERAL MATHEMATICS

GENERAL MATHEMATICS

Subject

General Mathematics

Credit value

3 credits

Prerequisites

SSC Mathematics

Course Description

This course is designed to prepare Student Teachers for teaching mathematics in elementary grades. It provides opportunities for Student Teachers to strengthen their mathematical knowledge and skills and to gain confidence in their understanding of mathematics. An important outcome of this course is for Student Teachers to be able to teach mathematics successfully in the primary, elementary, and middle grades.

Research-based knowledge about good maths instruction provides a solid base of information for educators to use as they identify mathematics skills that Student Teachers need to develop, as well as teaching strategies and instructional approaches that best support the development of these skills. The course design is based on what research tells us about good maths instruction.

The overall organization of the course is divided into four units:

1) Numbers and operations
2) Algebra
3) Geometry and geometric measurement
4) Information handling

Each unit of study has a consistent design or organization and is meant to maximize Student Teachers' time for learning.

Content

Most one-hour sessions will begin by working on a maths problem. Student Teachers will engage in solving and discussing a maths problem and sharing approaches and solutions. The content has been developed so that Student Teachers will engage in mathematics in depth to help them connect concepts within and across the four units.

Pedagogy

In each lesson, Student Teachers will actively engage in doing mathematics in order to experience approaches to teaching and learning maths that they can use when they teach. They will recognize that there are often multiple ways of approaching a problem and, in some instances, more than one correct answer. The Instructor will present questions that stimulate curiosity and encourage Student Teachers to investigate further by themselves or with their classmates.

The course will also examine how children learn and develop mathematical understanding and skills and how the way children think influences the teaching of mathematics in elementary grades.

Assignments

Student Teachers are expected to continue learning about maths and the teaching of maths after class. There will be assignments to stretch their content knowledge and to learn more about teaching maths. Assignments will take many forms, including independently solving maths problems and school-based tasks.

In summary, the General Mathematics course is a comprehensive effort to build and deepen maths content knowledge, learn and use high-quality instructional practices, and study ways in which children approach and learn mathematics.

Course objectives

Student Teachers will be able to:

- increase their mathematical content knowledge for numbers and operations, algebra and algebraic thinking, geometry and geometric measurement, and information handling for teaching in elementary grades
- increase their confidence, competence, interest, and enthusiasm for mathematics by exploring and doing mathematics
- deepen an understanding of how children learn mathematics
- build a variety of instructional techniques with clear purposes
- enhance their use of questioning techniques to elicit children's understanding
- learn ways to engage children in mathematical thinking through interactive activities.

Semester outline

UNIT 1:		Numbers and operations (5 weeks, 15 hours)
Week	**Themes**	**Subthemes**
1	Addition and subtraction Equivalence	Counting Models for addition and subtraction with natural numbers Addition and subtraction as inverse operations Word problems involving addition and subtraction
2	Place value Multiplication and division of whole numbers	Working in the base-10 system Models for multiplication with natural numbers Multiplication and division as inverse operations Models for division with natural numbers Nature of the remainder in division Factors, prime, and composite numbers
3	Fractions and decimals	Models of fractions (sets, number line, area, volume) Types of fractions (proper, improper, and mixed number) Decimals as fractions linked to base-10 place value Concept of GCF and LCM Operations with fractions and decimals
4	Per cent Ratios and proportion Rates	Per cent as related to fractions and decimals Ratio and proportion Rates
5	Integers	Integers, operations with integers Venn diagrams

The Student Teacher will be able to:

- differentiate between various types of numbers in our number system
- know various models for arithmetic operations (addition, subtraction, multiplication, and division) with natural numbers, rational numbers, and integers
- understand base-10 place value as it relates to natural numbers and eventually to decimals
- be able to describe the relationship among and between fractions, decimals, ratios, rates, proportions, and percentages.

UNIT 2: Algebra (4 weeks, 12 hours)

Week	Themes	Subthemes
1	Algebra as generalized arithmetic Patterns	Repeating patterns and growing patterns Generalizing a pattern and finding a rule
2	Algebraic terminology x as a variable Coordinate graphs Multiple representations Identity	Creating coordinate graphs Continuous, discontinuous, and discrete graphs Equivalent expressions
3	Linear functions Order of operations	Interpreting tables, graphs, and equations of linear functions The concept of slope Order of operations
4	Square expressions and equations Symbol manipulation	Interpreting tables, graphs, and equations of quadratic functions Solving for x, the unknown

The Student Teacher will be able to:

- describe the connection between arithmetic and algebra
- identify the repeating and/or increasing unit in a pattern and express that pattern as a rule
- understand what variables are and when and how variables are used
- express algebraic relationships using words, tables, graphs, and symbols
- use order of operations to solve for unknowns in algebraic equations.

UNIT 3: Geometry and geometric measurement (5 weeks, 15 hours)

Week	Themes	Subthemes
1	Polygons	Characteristics of polygons with an emphasis on triangles and quadrilaterals Benchmark angles
2	Undefined terms in geometry Identification and construction of angles	Point, line, line segment, and ray Models of angles Classifying angles by measurement Tessellations
3	Geometric measurement Area and perimeter of polygons and irregular shapes	Perimeter and area formulae
4	Geometric measurement Circumference and area of circles Surface area of cuboids and cylinders	Circumference and area formulae Surface area formulae
5	Volume of cuboids and cylinders Introduction to the Pythagorean theorem	Volume formulae Squares, square numbers, and square roots (surds) The Pythagorean theorem

The Student Teacher will be able to:

- understand undefined terms in geometry
- identify and construct different types of angles
- identify characteristics and measurable attributes of two-dimensional figures and three-dimensional objects
- calculate area, perimeter, surface area, and volume
- understand square numbers, square roots, and the relationships involved in the Pythagorean theorem.

UNIT 4: Information handling (2 weeks, 6 hours)

Week	Themes	Subthemes
1	Graphic displays of information	Collect and organize data via tally marks, pictographs, line plots, bar graphs, and line graphs (discrete and continuous) Interpret these graphic displays of data
2	Measures and central tendency	Range Mean Median Mode

The Student Teacher will be able to:

- recognize and construct various types of graphs
- determine which types of graphs best describe a given situation
- analyse a graph and interpret its information
- understand different measures of central tendency and determine which best describes a given situation.

Course grading policy

A variety of assessments will be used to assign a final grade. It is recommended that course work be used to assign at least 50 per cent of the final grade.

Your Instructor will tell you at the start of the course how your final grade will be determined and which pieces of course work will be assessed.

Suggested resources

These resources provide additional information about maths education and the mathematical topics addressed during the course. However, there are many other resources (books, websites, and articles) that you could use in addition to or instead of those listed below.

National Council of Teachers of Mathematics, 'Illuminations'.
➢ http://illuminations.nctm.org

New Zealand Ministry of Education, 'New Zealand Maths', Curriculum.
➢ http://nzmaths.co.nz

University of Cambridge. 'NRICH: Enriching Mathematics'.
➢ http://nrich.maths.org/public

Tom Bassarear, *Mathematics for Elementary School Teachers* (Belmont, CA: Brooks/Cole, 2012).

Suzanne Donovan, John Bransford, National Research Council, *How Students Learn: History, Mathematics, and Science in the Classroom* (Washington DC: National Academies Press, 2005).
➢ www.nap.edu/catalog.php?record_id=10126#toc

Derek Haylock, *Mathematics Explained for Primary Teachers* (Thousand Oaks, CA: SAGE Publications, 2001).

John A. Van de Walle, *Elementary and Middle School Mathematics: Teaching Developmentally* (Boston: Allyn and Beacon, 2004).

Nancy Protheroe, 'What Does Good Mathematics Instruction Look Like?'
➢ http://www.naesp.org/resources/2/Principal/2007/S-Op51.pdf

Planning guide

UNIT 1

NUMBERS AND OPERATIONS

FACULTY NOTES

Unit 1/week 1: Addition, subtraction, and equivalence

Session 1: Models for addition
Session 2: Models for subtraction
Session 3: Equivalence, thinking like children

Faculty preparation for the upcoming week (1–2 hours)

- Read this following article:
  - 'Addition and Subtraction in the Primary Grades':
    - ➢ http://tinyurl.com/Add-Subtr-Primary
- Look through the following websites that address addition, subtraction, equivalence, and the way children think about mathematics:
  - Young children's intellectual development when adding:
    - Counting from one:
      - ➢ http://tinyurl.com/Counting-From-One-NZ
    - Counting on:
      - ➢ http://tinyurl.com/Counting-On-NZ
  - Models of addition and subtraction:
    - Addition:
      - ➢ http://tinyurl.com/IllumAddition
    - Subtraction:
      - ➢ http://tinyurl.com/IllumSubtraction
  - The balance model for addition and subtraction using a ruler, pencil, and paperclips:
  - http://tinyurl.com/IllumBalance
  - Cognitively Guided Instruction (CGI):
    - ➢ http://tinyurl.com/CGI-Joining
    - ➢ http://tinyurl.com/CGI-Comparison
    - ➢ http://tinyurl.com/CGI-Separate
    - ➢ http://tinyurl.com/CGI-Joining-2
  - Decomposing numbers:
    - ➢ http://tinyurl.com/Decomp-Assess-NZ
  - Children's work samples:
    - Addition:
      - ➢ http://budurl.com/AdditionButterflies
    - Addition:
      - ➢ http://budurl.com/AdditionCats
    - Subtraction:
      - ➢ http://budurl.com/SubtractionFish
    - Subtraction:
      - ➢ http://budurl.com/SubtractionParrot

- The equals sign:
  - http://tinyurl.com/Balance-Discover-1
  - http://tinyurl.com/Balance-Cubes-1

- Download and print out the following handouts for Student Teachers:
  - 'Analysing Children's Thinking in Addition and Subtraction' (handout in Course Guide resources)
  - Counting from one:
    - http://tinyurl.com/Counting-From-One-NZ
  - Counting on:
    - http://tinyurl.com/Counting-On-NZ
  - Number line 0–24:
    - http://tinyurl.com/Number-Line-1-to-24
  - CGI Frameworks (three pages):
    - http://tinyurl.com/cgiFrameworks
  - Subtraction (children's work sample: take away, appropriateness of visual representation):
    - http://budurl.com/SubtractionCookies
  - Addition (children's work sample: near doubles):
    - http://budurl.com/AdditionCandy
- Read through the plans for this week's three sessions.

Weeklong overview

This unit begins with one equation ($5 + 7 = 12$) that Student Teachers will explore in depth during all three sessions this week.

During Session 1, the emphasis will be on something that most adults take for granted: simple addition.

This idea will echo throughout this entire course: although Student Teachers need to understand maths as adults, they also need to understand mathematics through the eyes of children.

This session is designed to challenge Student Teachers to think not about what they already know (such as number facts) but about how children might begin thinking about addition. Thus, as basic as it seems, the first topic discussed in Session 1 is the strategy young children use when beginning to add: counting, counting on, and even counting on their fingers as a readily available device for sums through 10.

The second topic for Session 1 is how a given number can be decomposed. For example, 12 can be expressed not only as $5 + 7$, but $7 + 5$ (which leads to children's becoming aware of the commutative property of addition at a very early age). But 12 can also be decomposed into doubles ($6 + 6$), more than two addends ($5 + 4 + 3$), and as a way to introduce place value ($10 + 2$).

The third topic for this session is that of four models for addition—joining sets, counting on, moving forward on a number line, and balance/equivalency—which will be developed more fully in the third session of this week.

Session 2 will build on what Student Teachers have just learned about models for addition to introduce models for subtraction. Besides adapting the four addition models given earlier for subtraction—separating a subset, counting back, moving backward on a number line, and balance/equivalency—a fifth model, comparison, is included. An example of this is 'I have three brothers and two sisters. How many more brothers than sisters do I have?' In this situation, children need to address one-to-one correspondence and see how many match up and how many are left over (which is the solution to the problem).

This session will end with a reading assignment for Student Teachers to do for homework. It is an introduction to Cognitively Guided Instruction, a rigorously researched method to organize the addition and subtraction models already discussed. Both articles will be used as starting points for discussion in Session 3.

Unit 1/week 1, session 1: Addition

What do Student Teachers need to know?

Most Student Teachers probably think of addition as joining sets. However, there are three other models: 1) counting, 2) using a number line, and 3) the balance model, which will be developed more fully in the third hour of this week, with an emphasis on the concept of equivalence. For more information on these different models, check out:

➢ http://tinyurl.com/IllumAddition

Decomposition of numbers. Young children's decomposition of a number into new combinations is the foundation for the associative and commutative properties of addition. It also allows children to build their number sense by creating alternative ways of thinking about a number.

How do children think about these concepts?

When thinking about the addition model of counting, young children move from 'counting from one' to 'counting on'. These two short handouts explain the difference in young children's intellectual development when adding:

- Counting from one:
  ➢ http://tinyurl.com/Counting-From-One-NZ
- Counting on:
  ➢ http://tinyurl.com/Counting-On-NZ

When using the number-line model, young children are often unsure if they should begin counting from 0 or 1. With sets they need to begin at 1, but when using a number line they need to begin at 0 and count 'jumps'. Eventually they will begin to notice that they can start at 5 and make 7 jumps (counting on) to arrive at 12.

When decomposing numbers, as in the case of 5 + 7 = 12, how might the 5 and the 7 be decomposed, then recomposed into new addends? Some children may be 'looking for 10' and create 5 + (5 + 2) = 12. Others may be thinking about 'doubles': (5 + 1) + 6 = 12. In either case, children are beginning to manipulate numbers and develop their number sense. Later on, they will feel comfortable decomposing two-digit numbers into tens and units, and eventually when working with multiplication, they will realize that they can decompose a number into its factors.

When asked to decompose a number (such as 12), young children often do so in a random manner: 8 + 4, 3 + 9. Although the teacher should record the children's responses as they are given, the next step is to enquire how their responses might be organized. Helping children rearrange their responses into an organized list will allow them to see a pattern, which will help them generalize a number's decomposition and build their number sense. Look at this website:

➢ http://tinyurl.com/Decomp-Assess-NZ

What is essential to do with Student Teachers?

- Introduce the four models for addition.
- Explain decomposing (and then recomposing) numbers.
- Relate each of these points to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers in pairs (or groups of no more than four) to brainstorm for five minutes about all the mathematics that may be implicit in the equation 5 + 7 = 12. Ask them to share their thoughts while you record the ideas on chart paper for future reference. After this activity, note with a check mark ideas that will be discussed during the week. Congratulate them for their willingness to go beyond their first impression and delve more deeply into the mathematics! (Add to the list any topics for the week that Student Teachers did not mention.)

Introduce the four models for addition: joining sets, counting on, hopping forward on a number line, and creating equivalence.

Give Student Teachers the following scenario and ask them to 'count on' by using their fingers (as young children would) to solve this problem: 'I get up early and eat breakfast at 5 in the morning; 7 hours later I eat lunch. When do I eat lunch?' Ask how 'counting on' is somewhat different from 'counting from 0'.

Distribute the number line handout and have Student Teachers use the number line to add 5 + 7. Notice which Student Teachers started from 0, hopped to 5, and then hopped 7 more places, and which of them began at 5 and 'counted on'. Relate these two different ways of modelling the problem to the 'starting from 0' and the 'counting on' methods. (Also note that young children are often unsure about whether they should begin counting from 0 or from 1.) Have Student Teachers think of a ruler as a model for a number line, one that includes not only whole numbers but also fractions or decimals. How might they use rulers to help children think about adding (and later subtracting)?

Introduce the concept of decomposing a number.

Have Student Teachers solve the problem: 'In my apartment building there are 12 cats and dogs. How many might there be of each?' Ask for random answers and record these on chart paper. When all combinations have been given, have Student Teachers create an organized list of the results in their notebooks. Discuss why helping young children organize mathematical information is an important step in noting patterns, not just for computation, but later on for patterns in algebra function tables ($a + b = 12$). Mention that they will use this chart in the next session to discuss subtraction.

Assignment

To be determined by the Instructor.

Unit 1/week 1, session 2: Subtraction

What do Student Teachers need to know?

Just as Student Teachers may have assumed that addition was limited to joining sets, they may consider subtraction only as taking away. As with addition, however, subtraction can be conceptualized in four other ways: 1) counting backward, 2) moving backward on a number line, 3) the balance model, which will be developed more fully in the third hour of this week, with an emphasis on the concept of equivalence, and 4) comparison. For more information on these different models, refer to 'Models for Subtraction':

➢ http://tinyurl.com/IllumSubtraction

Linking the operations of addition and subtraction allows Student Teachers to clarify their understanding of how these two operations are each other's inverse, which will undo what the other operation just did. (The concept of the inverse will also be developed further when multiplication and division are discussed.) This also would be a good time to review the charts Student Teachers created for the '12 cats and dogs' activity. How could the table, which originally charted addition, become a visual reference for subtraction?

How do children think about these concepts?

When using the comparison model for subtraction, children see the total number of items in both sets; for example, 11 paperclips: 3 large and 8 small. But they need to address the items in those sets via one-to-one correspondence to discover which set contains more, which contains fewer, and what the difference is between the two. Again consider the '12 cats and dogs' activity from the vantage point of the question: 'How many more?' I have 7 dogs and 5 cats. Of which do I have more? How many more? The answer in the comparison model, although numerically the same, is substantively different from the remainder, which is the answer to the 'taking away' model. (There were 7 birds on a tree. Four flew away. How many remained?)

What is essential to do with Student Teachers?

- Introduce the five models for subtraction.
- Relate subtraction to the decomposing (and then recomposing) of numbers that Student Teachers performed in the prior session on addition.
- Link subtraction to addition by discussing how these two operations are inverses that undo each other.
- Relate each of these points to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers to solve the equation $8 - 3 = ?$ by counting backward. Suggest that they solve this problem as young children (who do not know the answer) would by asking them to count backward to find the solution: using their fingers. Notice which Student Teachers decompose the 8 into 4 fingers on each hand and which decompose the 8 into 5 fingers on one hand and 3 on the other.

Have Student Teachers recall the models for addition and introduce their counterparts in subtraction: 1) joining sets (taking away a set), 2) counting forward (counting backward), and 3) hopping forward on the number line (hopping backward).

Introduce the idea that these three models highlight the idea of addition and subtraction being inverse operations, in which addition undoes subtraction and vice versa.

Have Student Teachers experiment with the number line model for subtraction, 'hopping backward' from the starting number (the minuend) to find the answer to 8 – 3 = ?

Have Student Teachers refer to the organized chart they created for the problem: 'In my block of flats there are 12 cats and dogs. How many might there be of each?' Ask how their chart, which was used to model addition, can be used to describe subtraction.

Pose the following questions: 'If there are 7 cats, then the remainder are dogs. How many are dogs? If there are 5 cats, then the remainder are dogs. How many are dogs?' Mention to Student Teachers the use of the word *remainder*, which implies separating a total quantity into two sets.

Use this work with the set model to introduce another subtraction model (which did not have a corresponding model in the 'addition' session): comparison.

Use the two-column organized chart from the prior session, adding a column to the right labelled 'Difference', to begin working with the subtraction model of comparison.

12	12	Difference
0	12	12 – 0 = 12
1	11	12 – 1 = 11
2	10	12 – 2 = 10
3	9	12 – 3 = 9
4	8	12 – 4 = 8
5	7	12 – 5 = 7
etc.		

As Student Teachers consider the chart, ask, 'Which has more? How many more?' as you add entries into the 'Difference' column.

After the chart is complete, ask Student Teachers:

- Do you see any patterns as you look at the differences?
- Is there a symmetrical pattern? If so, why?
- Is there an odd or even pattern? If so, why?
- What does the chart show about the commutative property of addition?
- What does the chart show about the inverse operations of addition and subtraction?

To end the session, have Student Teachers consider the numbers 13, 7, and 6, and have them create two real-life subtraction scenarios:

- One in which the subtraction model is 'take away' and results in a remainder.
- The other in which the scenario involves a comparison and results in a difference.

Assignment

To be determined by the Instructor.

Unit 1/week 1, session 3: Equivalence, thinking like children

What do Student Teachers need to know?

The notion of equivalence is one of the major, overarching concepts in all of mathematics. However, both adults and children often misinterpret the equals sign in an equation as meaning 'the answer is …' rather than understanding it to be an indicator of the important mathematical notion that there is a balance, or *equivalence*, on each side of the equation.

Cognitively Guided Instruction (CGI) word problems help teachers codify children's understanding of addition, subtraction, and the relationship between these two operations. Recall the various models of addition and subtraction that were discussed in the prior sessions. Then consider the equation $3 + 4 = 7$. Here are several ways this equation could be translated into CGI word problems:

- I have 3 raisins and get 4 more. How many do I have now? (This would be an example of a joining problem in which the **result** [7] is unknown.)
- I have 3 raisins and get some more. Now I have 7 raisins. (This would be an example of a joining problem in which the **change** [+4] is unknown.)
- I have some raisins. I get 4 more and now have 7. (This is the third type of joining problem, only in this case the **start** [3] is unknown.)

One of the most useful ways of assessing Student Teachers' understanding is analysing their work samples.

How do children think about these concepts?

Especially important is a comment about word problems in the article found at http://tinyurl.com/CGI-Joining: 'This type of problem illustrates a difference between child and adult thinking. Adults would identify this as a subtraction problem, but children do not. Children see this as a problem requiring a joining action'.

Misinterpretation about the equals sign is why many young children looking at the open equation $3 + ? = 7$ immediately conclude that the answer is 10. They see a 3, a 7, a plus sign, an equals sign, and then mentally rearrange those four elements into the more familiar $3 + 7 = ?$, where the answer is indeed 10.

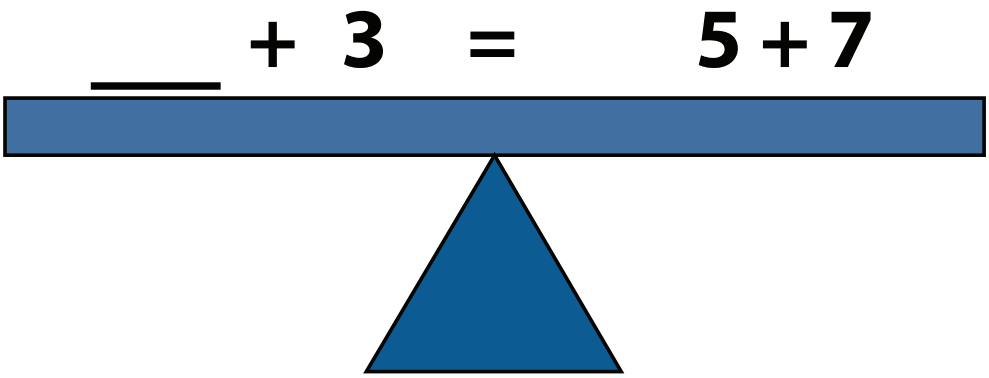
Until children have a clear understanding of number sense combined with operations sense, we can expect them to assume that the answer for the '?' in the equation $3 + ? = 7$ is probably 10.

Student Teachers need to realize that it takes time, several models, and many hands-on experiences to help their future primary grade students coordinate number sense, operations sense, and computational fluency until they finally realize that for the equation $3 + ? = 7$, the unknown is 4, not 10.

The following websites include the balance model for addition and subtraction and describe how teachers can guide children's thinking toward understanding equivalence across the equals sign:

- http://tinyurl.com/Balance-Discover-1
- http://tinyurl.com/Balance-Cubes-1

If a commercial balance is not available, the concept of balanced equations can be described to children as young as six by having them consider a teeter-totter or see-saw. Have children relate this visual model to an open addition sentence, such as:



Children need to make generalizations about addition, subtraction, and equivalence that apply to many situations. This is why having several different models for these operations of addition and subtraction is important. However, the teacher's role is not only to introduce these models but also to show how they are related to each other and how they are connected to the inverse operation.

Children eventually need to develop computational fluency, recalling their number facts with automaticity—which is not the same as memorization. For example, when asked for the sum of $5 + 6$, a child may not have the answer (11) memorized, but he or she may quickly use the concept of 'near doubles', recalling that because $5 + 5$ equals 10, $5 + 6$ must equal 11. Similarly, for any equation that involves subtracting 2, the child may quickly use 'counting back 2' or visualizing hopping back 2 spaces on the number line. Thus, a child who has internalized these strategies and has learned about place value should be able to find the answer to $35 – 2 = ?$ Perhaps even more important, they can use this strategy to solve equations that bridge a decade, such as $31 – 2 = ?$, without resorting to regrouping.

What is essential to do with Student Teachers?

- Introduce the concept of equivalence by using the model of a balance. Refer to this website:
  - http://tinyurl.com/IllumBalance
- Clarify the meaning of the equals sign.
- Review and clarify participants' understanding of Cognitively Guided Instruction after they have read for homework about the different models for result, change, and start.
- Have Student Teachers compare and analyse two children's work samples as they begin to assess children's understanding of addition and subtraction.

Activities with Student Teachers

Begin by asking Student Teachers to describe their understanding of the equals sign. What does it mean to them? What do they think the equals sign means to young children? Give two examples: $5 + 7 = ?$ and $5 + ? = 12$. Note that for adults, these equations look the same. But this is not necessarily true for children. How do Student Teachers think children will solve these equations?

Introduce the balance model for addition and subtraction, demonstrating $2 + 3 = ?$ by drawing a balance beam such as the shown earlier on chart paper or the board. Next, draw the balance model for the equation $2 + ? = 5$. Ask Student Teachers how they could balance both sides of the equation.

Distribute copies of the Cognitively Guided Instruction (CGI) Frameworks. Ask Student Teachers to describe their understanding of the different CGI models for addition and subtraction (result, change, start), asking their classmates to help clarify areas of confusion. Ask them about page three of the handout, where the difficulty level of different types of problems are rated. How does this relate to the different models of addition and subtraction they have studied this week?

Have Student Teachers work in pairs or small groups using the basic equation $5 + 7 = 12$ to generate word problems for result, change, and start. Make sure they create a story that illustrates comparison. Have several Student Teachers share their word problems, explaining why each story and its equation fit a particular CGI category.

End the session by distributing the handouts of the two children's work samples: 1) subtraction (take away, appropriateness of visual representation) and 2) addition (near doubles). Have Student Teachers work in groups of three or four to compare, analyse, and discuss what they perceive about the children's understanding. Ask them to consider: which models of addition and subtraction are being used? What can they infer about the child's understanding? How would they compare these two work samples with regard to mathematical sophistication?

Assignment

The handout 'Analysing Children's Thinking in Addition and Subtraction' asks Student Teachers to consider children's thinking about addition and subtraction in depth, going beyond whether they have reached the right or wrong answer. These questions allow for formative assessment, noticing what needs to be done to help a particular child or the class as a whole. The list also serves as a model for how teachers can develop their own lists of questions for other mathematical topics, such as multiplication, fractions, area and perimeter, etc.

Review the following four samples of student work:

- Addition (joining sets, figurative not realistic drawings):
  - ➢ http://budurl.com/AdditionButterflies
- Addition (joining sets, two items grouped to reflect the story, not all items in set are shown):
  - ➢ http://budurl.com/AdditionCats
- Subtraction (take away, erasures, appropriateness of story's context):
  - ➢ http://budurl.com/SubtractionFish
- Subtraction (take away, many drawings, additional note at the bottom):
  - ➢ http://budurl.com/SubtractionParrot

Ask Student Teachers to consider the following:

- Which addition and subtraction models are being used?
- What can you infer about each child's understanding?
- How would you rank these samples with regard to mathematical sophistication?

FACULTY NOTES

Unit 1/week 2: Place value in base 10, multiplication and division of whole numbers

Session 1: Place value for whole numbers in base 10
Session 2: Multiplication of whole numbers
Session 3: Division of whole numbers

Faculty preparation for the upcoming week (1–2 hours)

- Review the following websites that address place value, multiplication, and division:
  - Multiplication:
    - http://tinyurl.com/ThinkMath-Mult
  - Multiplication and division:
    - http://tinyurl.com/ThinkMath-Mult-Div
  - Multiplication:
    - http://tinyurl.com/ThinkMath-MultVsAdd
  - Multiplication and repeated addition:
    - http://tinyurl.com/Mult-Repeat-Add
  - Multiplication and division as inverses (video):
    - http://tinyurl.com/Relate-Mult-Divi
- Download and print out for Student Teacher use:
  - Number line 0–24:
    - http://tinyurl.com/Number-Line-1-to-24
  - Ten-frame mats:
    - http://tinyurl.com/TensFrames
  - Hundred Chart:
    - http://tinyurl.com/Chart-100
  - Multiplication table:
    - http://tinyurl.com/Mult-Chart-144
- Bring to class:
  - A package of beans to use as counters
  - Graph paper
- Read through the plan for this week's three sessions.

Weeklong overview

Session 1 begins by addressing place value by using the equation from last week, 5 + 7 = 12. Session 1 then uses another equation (6 x 3 = 18) that Student Teachers explore in depth to study the concepts of factors, multiples, multiplication, and division.

During the first session, Student Teachers will focus on three things: 1) place value, 2) visual models for understanding place value, and 3) thinking about mathematical strategies for solving equations.

Mathematical strategies for solving equations include the following three issues to emphasize:

- Although all equations are equal, not all equations are equally useful as teaching tools. Selecting the equation $5 + 7 = 12$ for exploration last week was a conscious instructional decision because the equation connects to the concept of place value this week. This connection indicates how teachers need to consider the implications of the numbers they use when using exemplars to introduce maths topics.
- Many maths problems have one right answer, such as $5 + 7 = 12$. There are, however, different strategies that children might use to arrive at that one right answer. Some of their strategies may seem more or less efficient to us adults. But efficiency is an end goal, once mathematical understanding is in place.
- Valuing and discussing alternative solution strategies is an important way to help students make mathematical connections and see mathematical equivalency among different solution strategies. (This will become even more evident when considering proofs in algebra and geometry, which is why it is important to include discussions on alternative solution methods even at the beginning of the 'Numbers and Operations' unit.)

Session 2 will address multiplication of whole numbers. Just as we began last week with a single equation to be explored in depth for its potential regarding addition and subtraction, we will do the same for multiplication and division, using the basic equation $6 \times 3 = 18$. This session will introduce several models for multiplication: array, intersections, area, partial products, and skip-counting.

Session 3 will introduce division of whole numbers as the inverse of multiplication. (This parallels the idea of addition and subtraction as inverse operations that undo each other.) Models for division of whole numbers will be introduced, and the issue of the remainder will be discussed as it relates to a problem's context.

Unit 1/week 2, session 1: Place value in base 10

What do Student Teachers need to know?

Although all equations are equal, not all equations are equally useful as teaching tools. Selecting the equation $5 + 7 = 12$ for exploration last week was a conscious instructional decision, because the equation connects to the concept of place value this week. This connection indicates how teachers need to consider the implications of the numbers they use when using exemplars to introduce maths topics.

Many maths problems have one right answer, such as $5 + 7 = 12$. There are, however, different strategies that children might use to arrive at that one right answer. Some of their strategies may seem more or less efficient to us adults. But efficiency is an end goal, once mathematical understanding is in place.

Valuing and discussing alternative solution strategies is an important way to help students make mathematical connections and see mathematical equivalency among different solution strategies. (This will become even more evident when considering proofs in algebra and geometry, which is why it is important for children to discuss alternative solution methods even in the 'Numbers and Operations' unit.)

When young children need to go beyond the number 9 in counting and addition, they move toward multi-digit number sense, encountering the concept of tens and units.

There is a major difference between digits and numbers. For example, the '1' in the number 13 is simply a digit representing the number '10'.

Young children can begin working with place value by using handfuls of small objects such as pebbles or beans and arranging them into groups of 10, with perhaps some left over. At some point, however, the number of physical items becomes unwieldy, suggesting other models would be better, more efficient ways to further concept development.

To help children develop multi-digit number sense, two simple Ten Frames can be used to model why we need to use two digits to describe the number 12 in our base-10 number system, as each frame can only accommodate 10 items.

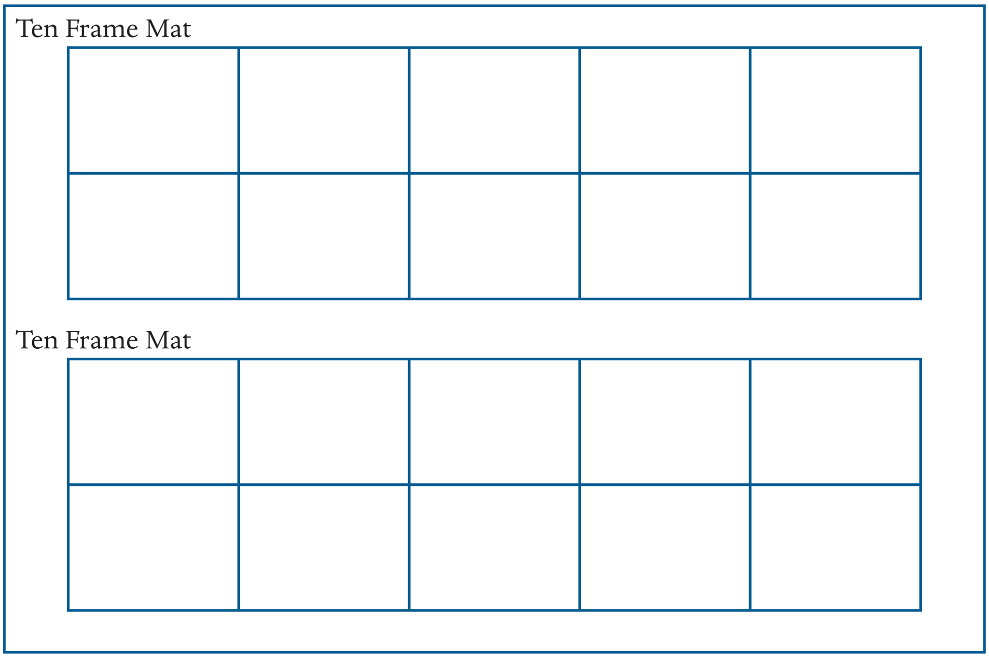


Figure: Ten frame mat

The number-line model, used earlier for simple addition, allows children to 'count on' once they reach the end of a decade, such as when adding 28 + 3. Without much effort, children will discover that they have moved from the 20s into the 30s. This model helps children avoid the assumption that they need to use an algorithm to solve this type of 'bridging the decade' problem.

A Hundred Chart (which is really ten 10-place number lines stacked upon each other) allows children to model numbers from 1 to 100 (or 0 to 99). The Hundred Chart also permits children to not only move back and forth across rows to add and subtract units, but also to move up and down columns to add or subtract 10s.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure: Hundred chart

When working with a Hundred Chart, teachers need to decide if they will use one that begins with 0 and goes to 99 (each row on the chart beginning with the same 'tens number': 0–9, 10–19, 30–39, 90–99) or one that begins with 1 and goes through 100, with each row ending with the next 10: 1–10, 11–19, 31–40, 91–100.

The basic number sentence $5 + 7 = 12$ can also be transformed into 5 tens + 7 tens = 12 tens, or $50 + 70 = 120$, moving beyond 100. That same $5 + 7 = 12$ equation can also be used as a way for children to use patterns to use mental addition to solve $25 + 7$, $35 + 7$, or $65 + 7$, all without using a formal algorithm.

How do children think about these concepts?

When teaching children about place value in base 10, it is not only important to use a variety of physical and visual models, but also to help children make connections among those various models. Sorting a pile of beans into groups of 10 with leftovers (units) may come first. Next, small sticks, straws, or toothpicks can be bundled into groups of 10 with a rubber band.

Once children have become comfortable with physical models, they can use the Ten Frames as a way to organize items where there is 10 with some left over (e.g. $5 + 7 = 10 + 2$). They do this first by actually placing small items on the squares in Ten Frames, and then they move from the real to the representational by putting dots into the Ten Frames to represent the 12 items.

This is a key opportunity to highlight that when children 'bridge a decade', as in $5 + 7 = 12$, there is a developmental sequence that occurs in children's thinking when they are given an equation containing numbers and symbols: 1) a CGI-type story context that explains the equation ('There were 5 ducks in a pond, and there were 7 ducks on the shore. How many ducks were there in all?'), 2) the physical (the beans), 3) the representational (placing dots in Ten Frames), and 4) once again the symbolic ($5 + 7 = 12$).

More importantly, this sequence of how children think about, understand, and learn mathematics will be repeated not only in 'Numbers and Operations', but in other areas of maths, especially algebra: 1) narrative, 2) physical, 3) representational (which will later include tables and graphs), and finally 4) the symbolic.

After children become familiar with the one-dimensional linear model of the number line from 0 through 20, they can begin to transfer this visualization to the two-dimensional Hundred Chart, which is simply ten number-line segments reorganized into a stacked, more compact, manageable format.

Children learn not only what they are taught, but also what they see in their classroom. This is why it is important to have a number line from 0 through 100 (and later, from 20 to 120) prominently displayed in the room. The same is true for a Hundred Chart. These are mathematical references that children can use on a daily basis to solve maths problems. But more importantly, visuals seen every day will help develop children's number sense, transforming those visual models into intellectual models that will become a child's internal mental reference for understanding and working with mathematics.

What is essential to do with Student Teachers?

- Introduce the base-10 number system for whole numbers, linking this to the concept of decomposing two-digit numbers such as 12 into 10s with additional units.
- Provide Student Teachers with three visual models to help them think about how children can understand place value: the number line, Ten Frames, and the Hundred Chart.
- Relate each of these points to children's thinking.
- Mention that the base-10 number system does not relate only to whole numbers but that it extends to negative numbers and decimals, both of which will be discussed later in the unit.

Activities with Student Teachers

To introduce the concept of place value in our base-10 number system, distribute copies of the Ten Frames and a handful of beans. Explain the Ten Frame model and then have Student Teachers work in pairs to model the equation $5 + 7 = 12$. After they have done this, have a class discussion to discover how Student Teachers used the Ten Frames to solve the problem.

Anticipate that some Student Teachers may have placed 5 beans on one frame, 7 beans on another, and then moved 5 of the beans from the second frame onto the first, which resulted in one frame with 10 beans and another with 2.

Other Student Teachers may have used the counting principle by placing 5 beans on the first frame, then taking 7 beans and distributing 5 of them onto the first frame until it was full. At that point, they would have placed the remaining 2 beans onto the second frame, showing that the sum was 12.

Ask if there were other strategies that Student Teachers used. Ask if there was disagreement among partners as to which strategy was the right one. Note that both of the strategies were valid, but each is subtly different.

After they have explored ideas for 5 + 7 = 12, have them work in pairs with Ten Frames to find equations with sums that equal 11 through 20. Challenge Student Teachers to include not only single digit addends (such as 8 + 3) but equations where one addend has two digits (such as 13 + 6).

In a whole-class discussion ask how, for example, they modelled 13 + 6, and what the digits 1 and 3 mean for the number 13 and how the digits 1 and 3 relate to their Ten Frames.

Help Student Teachers understand that digits are representations for parts of a number. Thus the digit 1 in the number 13 means the number '10', not the number '1'. This distinction between digits and number is a crucial part of children's developing number sense. This distinction of digits versus numbers will have major implications when children begin to work with multi-digit calculations.

Introduce the Hundred Chart and have Student Teachers articulate patterns they see. If no one mentions that the Hundred Chart is really a series of number-line segments stacked upon one another, bring their attention to this.

Have Student Teachers work with the Hundred Chart to add multiples of 10 to a given number, perhaps starting with the number 7 and moving down the columns to model 7 + 10, 7 + 20, etc. What do Student Teachers notice about the patterns? How would they use the columns and rows in the Hundred Chart if they wanted to add 7 + 22? How is that different from adding 7 + 23 or 7 + 24 (where the sum moves into the next decade)?

Ask Student Teachers what this pattern would look like on a continuous number line from 1 to 100 where the sum moved to the next decade, such as with 7 + 23 becoming 30, or 7 + 24 becoming 31.

Finally, note the first item in the 'What do Student Teachers need to know?' section: that whereas all equations are equal, not all equations are equally useful as teaching tools. Ask how selecting 5 + 7 = 12, then 7 + 10, 7 + 20, 7 + 22, and finally 7 + 23 and 7 + 24 built mathematical understanding better than using random equations.

Point out that the idea of patterns will be developed more fully in the algebra unit, but that young children's work with patterns in numbers and operations is crucial to creating a coherent idea of mathematics as a logical system of thinking. Emphasize that this is why teachers need to consider the implications of selecting specific numbers when creating exemplars to introduce maths topics.

End by mentioning that base-10 place value applies not only to whole numbers but to negative numbers, decimals, percentages, exponents, etc. Base-10 place value is a concept children will meet again and again as they learn mathematics.

Assignment

To prepare for the next class, which is about multiplication, have Student Teachers read the two articles at these webpages:

- http://tinyurl.com/ThinkMath-Mult
- http://tinyurl.com/ThinkMath-Mult-Div

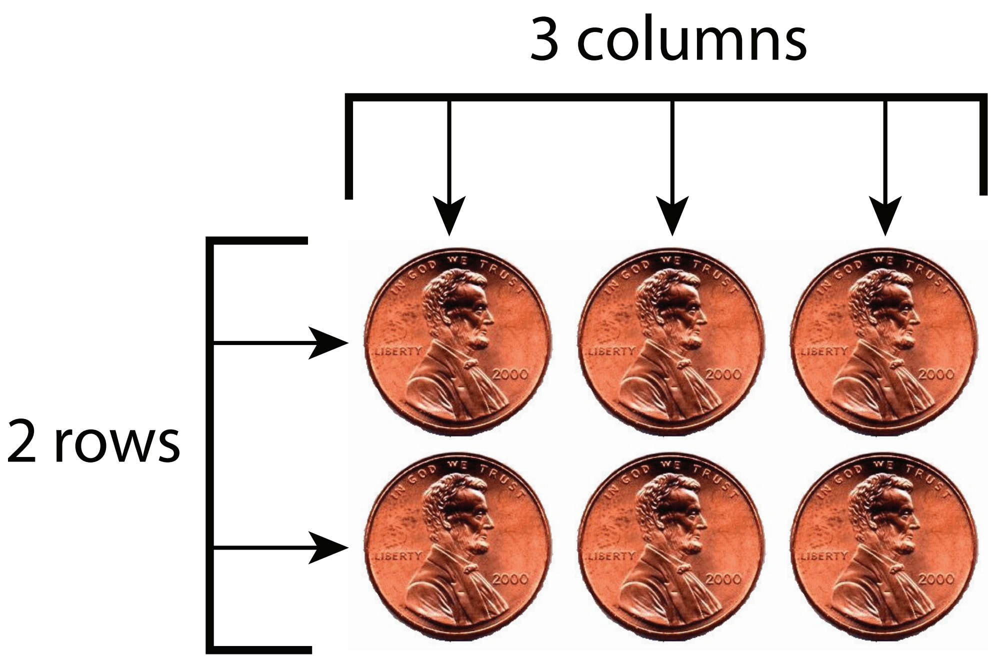
Unit 1/week 2, session 2: Multiplication of whole numbers

What do Student Teachers need to know?

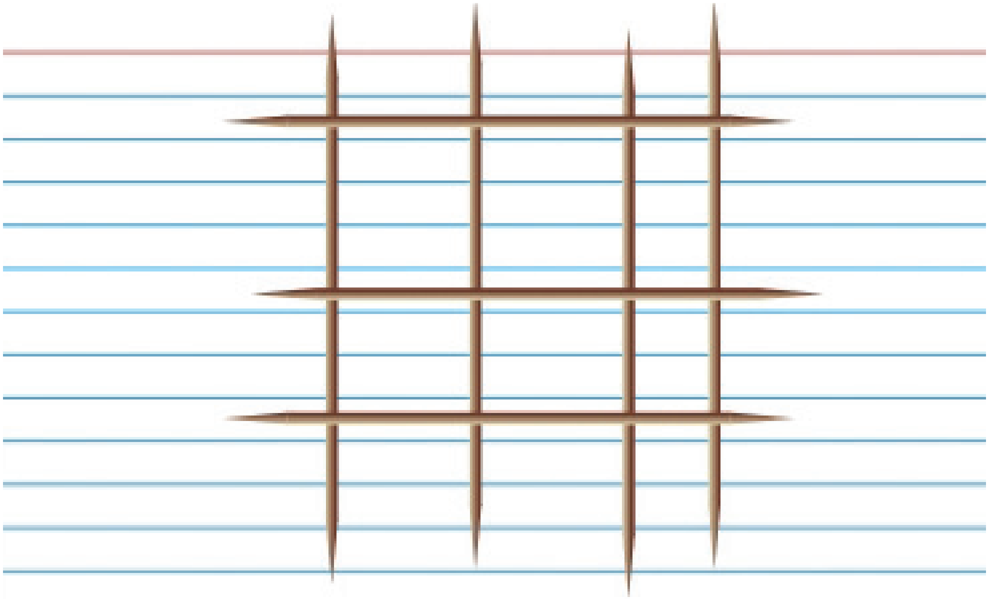
Just as addition is more than joining sets and subtraction is more than taking away, so, too, multiplication is more than repeated addition. Nor does multiplication always 'make things bigger'. This is one of the most important concepts to emphasize to Student Teachers this week: that what is implied by the multiplication of whole numbers does not hold true for integers, fractions, and decimals.

There are several models for multiplication, just as there were for addition and subtraction.

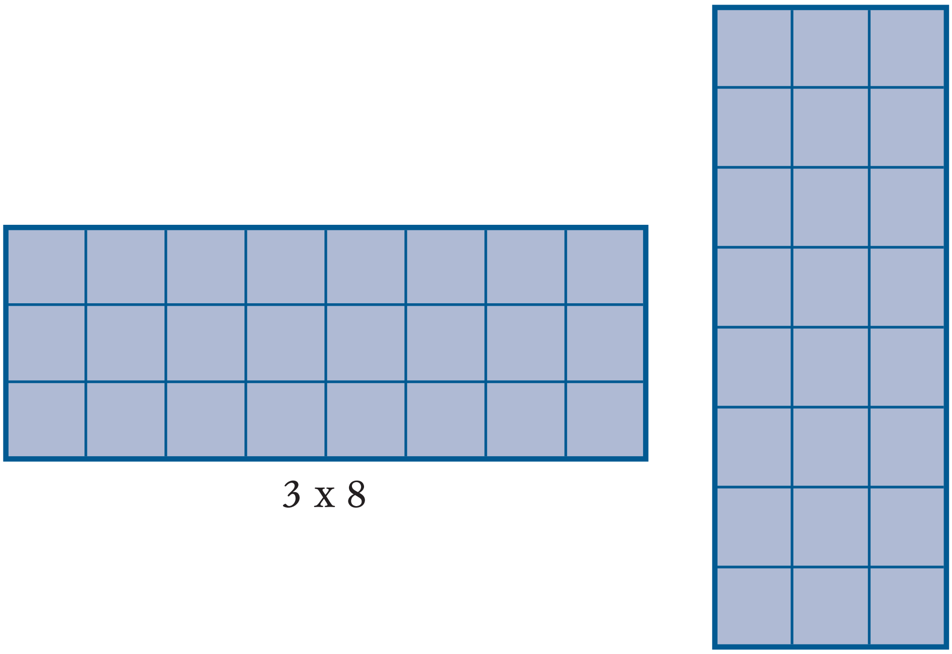
Array. This is essentially a set model for multiplication, arranging discrete items in rows and columns and then counting to find the answer. Most Student Teachers will be familiar with this model.



Intersections. This model, which can be thought of as streets and intersections, uses lines to show multiplication. In the intersection model, the lines represent the factors (in this case 4 and 3) and the sum of the intersections (12) is the product. (This will probably be a new model for both adults and children, and usually they will have to do several drawings to become convinced that this really works.)



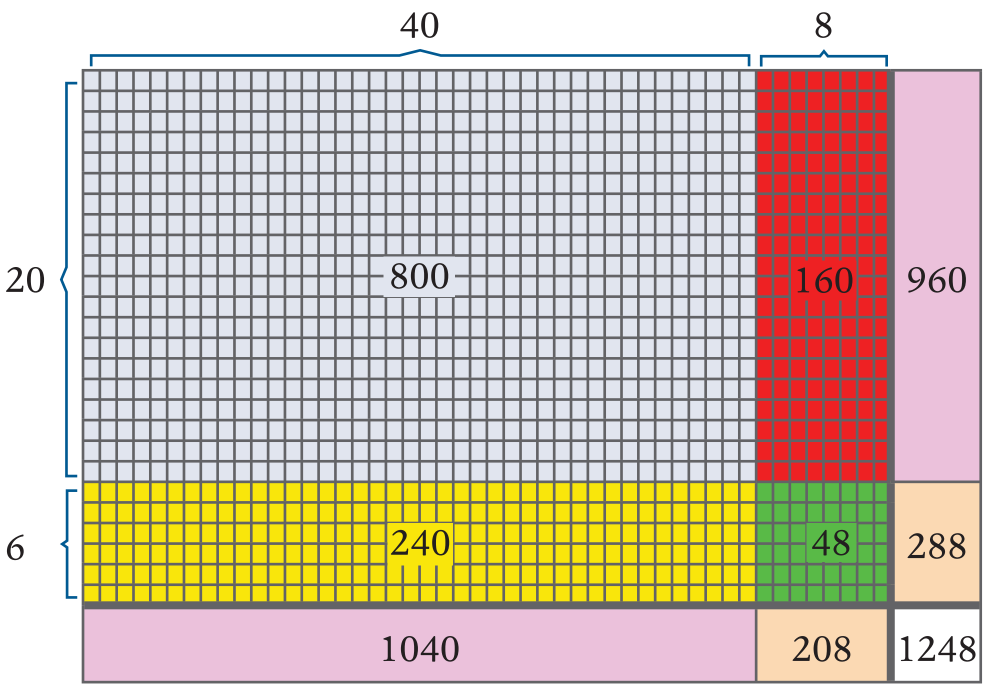
Area. This model sets the stage for several important concepts, including geometric measurement, multiplication of fractions, and prime and composite numbers. Note that this is different from an array as shown above in that the area model is continuous space, bounded by the rectangle, whereas the array was composed of discrete objects.



3 x 8

8 x 3

This second area model, used for multi-digit multiplication and called the 'partial products' model, builds on the simple area model described earlier, but now the factors are decomposed into tens and units. This will also be used as a visual model for multiplication in algebra. Although this picture shows the exact number of cells for the problem (indicating the magnitude of the partial products and solution), a simple hand-drawn set of labelled rectangles can suffice once Student Teachers understand the process.



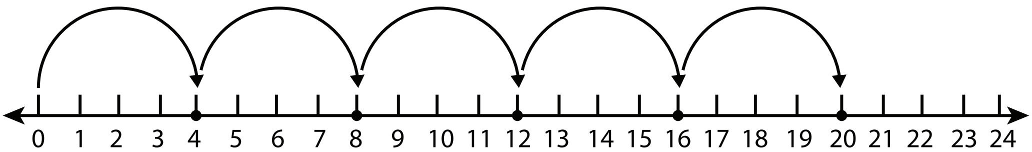
How do children think about these concepts?

Multiplicative thinking is substantively different from additive thinking, and it develops over time. At first, most young children think of multiplication as repeated addition for sets (two eyes per person, four legs per elephant, etc.).

However, this 'multiple sets' model is cognitively interesting, as 4 legs (per elephant) times 2 elephants does not result in 8 elephants but 8 elephant legs. For young children, this is conceptually different from addition and subtraction when they were working with sets with the same attribute. If you begin with 6 elephants and add 2 elephants, you have 8 elephants. Or, if you have 8 elephants and 2 elephants walk away, you have 6 elephants. Thus, when adding or subtracting the result will be ... elephants.

When multiplying, however, instead of dealing with one variable (elephants), children need to hold two variables (the number of elephants and the number of legs per elephant) in mind at the same time.

Another way that young children begin to experience multiplication in a purely numeric sense is by 'skip-counting' (2, 4, 6, 8 … or 5, 10, 15, 20 …), which can be modelled either by hops on the number line (in this case by 4s):



or by colouring cells representing those multiples on the Hundred Chart:

2

0	1	**2**	3	**4**	5	**6**	7	**8**	9
10	11	**12**	13	**14**	15	**16**	17	**18**	19
20	21	**22**	23	**24**	25	**26**	27	**28**	29
30	31	**32**	33	**34**	35	**36**	37	**38**	39
40	41	**42**	43	**44**	45	**46**	47	**48**	49
50	51	**52**	53	**54**	55	**56**	57	**58**	59
60	61	**62**	63	**64**	65	**66**	67	**68**	69
70	71	**72**	73	**74**	75	**76**	77	**78**	79
80	81	**82**	83	**84**	85	**86**	87	**88**	89
90	91	**92**	93	**94**	95	**96**	97	**98**	99

3

0	1	2	**3**	4	5	**6**	7	8	**9**
10	11	**12**	13	14	**15**	16	17	**18**	19
20	**21**	22	23	**24**	25	26	**27**	28	29
30	31	32	**33**	34	35	**36**	37	38	**39**
40	41	**42**	43	44	**45**	46	47	**48**	49
50	**51**	52	53	**54**	55	56	**57**	58	59
60	61	62	**63**	64	65	**66**	67	68	**69**
70	71	**72**	73	74	**75**	76	77	**78**	79
80	**81**	82	83	**84**	85	86	**87**	88	89
90	91	92	**93**	94	95	**96**	97	98	**99**

4

0	1	2	3	**4**	5	6	7	**8**	9
10	11	**12**	13	14	15	**16**	17	18	19
20	21	22	23	**24**	25	26	27	**28**	29
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60	61	62	63	**64**	65	66	67	**68**	69
70	71	**72**	73	74	75	**76**	77	78	79
80	81	82	83	**84**	85	86	87	**88**	89
90	91	**92**	93	94	95	**96**	97	98	99

5

0	1	2	3	4	**5**	6	7	8	9
10	11	12	13	14	**15**	16	17	18	19
20	21	22	23	24	**25**	26	27	28	29
30	31	32	33	34	**35**	36	37	38	39
40	41	42	43	44	**45**	46	47	48	49
50	51	52	53	54	**55**	56	57	58	59
60	61	62	63	64	**65**	66	67	68	69
70	71	72	73	74	**75**	76	77	78	79
80	81	82	83	84	**85**	86	87	88	89
90	91	92	93	94	**95**	96	97	98	99

6

0	1	2	3	4	5	**6**	7	8	9
10	11	**12**	13	14	15	16	17	**18**	19
20	21	22	23	**24**	25	26	27	28	29
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50	51	52	53	**54**	55	56	57	58	59
60	61	62	63	64	65	**66**	67	68	69
70	71	**72**	73	74	75	76	77	**78**	79
80	81	82	83	**84**	85	86	87	88	89
90	91	92	93	94	95	**96**	97	98	99

7

0	1	2	3	4	5	6	**7**	8	9
10	11	12	13	**14**	15	16	17	18	19
20	**21**	22	23	24	25	26	27	**28**	29
30	31	32	33	34	**35**	36	37	38	39
40	41	**42**	43	44	45	46	47	48	**49**
50	51	52	53	54	55	**56**	57	58	59
60	61	62	**63**	64	65	66	67	68	69
70	71	72	73	74	75	76	**77**	78	79
80	81	82	83	**84**	85	86	87	88	89
90	91	92	93	94	95	96	97	**98**	99

8

0	1	2	3	4	5	6	7	**8**	9
10	11	12	13	14	15	**16**	17	18	19
20	21	22	23	**24**	25	26	27	28	29
30	31	**32**	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	**48**	49
50	51	52	53	54	55	**56**	57	58	59
60	61	62	63	**64**	65	66	67	68	69
70	71	**72**	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	**88**	89
90	91	92	93	94	95	**96**	97	98	99

9

0	1	2	3	4	5	6	7	8	**9**
10	11	12	13	14	15	16	17	**18**	19
20	21	22	23	24	25	26	**27**	28	29
30	31	32	33	34	35	**36**	37	38	39
40	41	42	43	44	**45**	46	47	48	49
50	51	52	53	**54**	55	56	57	58	59
60	61	62	**63**	64	65	66	67	68	69
70	71	**72**	73	74	75	76	77	78	79
80	**81**	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	**99**

10

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

What is essential to do with Student Teachers?

- Introduce several models that can help Student Teachers visualize multiplication of whole numbers.
- Explain that multiplication is more than repeated addition or skip-counting.
- Have Student Teachers consider the fact that although multiplication of whole numbers makes the product greater than its factors, that this is not true for fractions, decimals, and integers.

Activities with Student Teachers

Begin the session by asking Student Teachers to consider the equation 6 x 3 = 18. Have them work in pairs to quickly write down all the mathematics they find implicit in this equation. Have them share their thoughts, charting their responses and using the terms *factors* and *products* as you work with their ideas.

Introduce the array or set model. Have Student Teachers draw 6 sets of 3 in an array, then have them create an array showing 3 sets of 6 in a different orientation. In both cases, the factors are 3 and 6 and the product is the same: 18. Ask how the two arrays are different. Ask when this difference in orientation might matter. (For example, if they needed to arrange 18 chairs in a narrow room in their home, they might create an arrangement of 6 short rows, each with 3 chairs. But if they were arranging 18 chairs in a classroom, they might decide to arrange the chairs in 3 rows, each with 6 chairs.) This is an opportunity to introduce and discuss the commutative property of multiplication and remind Student Teachers of the commutative property of addition.

Introduce the intersection model for multiplication by having Student Teachers draw a tic-tac-toe (noughts and crosses) grid. How many lines did they draw? Next, ask them not to focus on the cells (where they would they would enter an X or O) but on the intersections of the lines. How many intersections are there? How does this number relate to the lines that they drew?

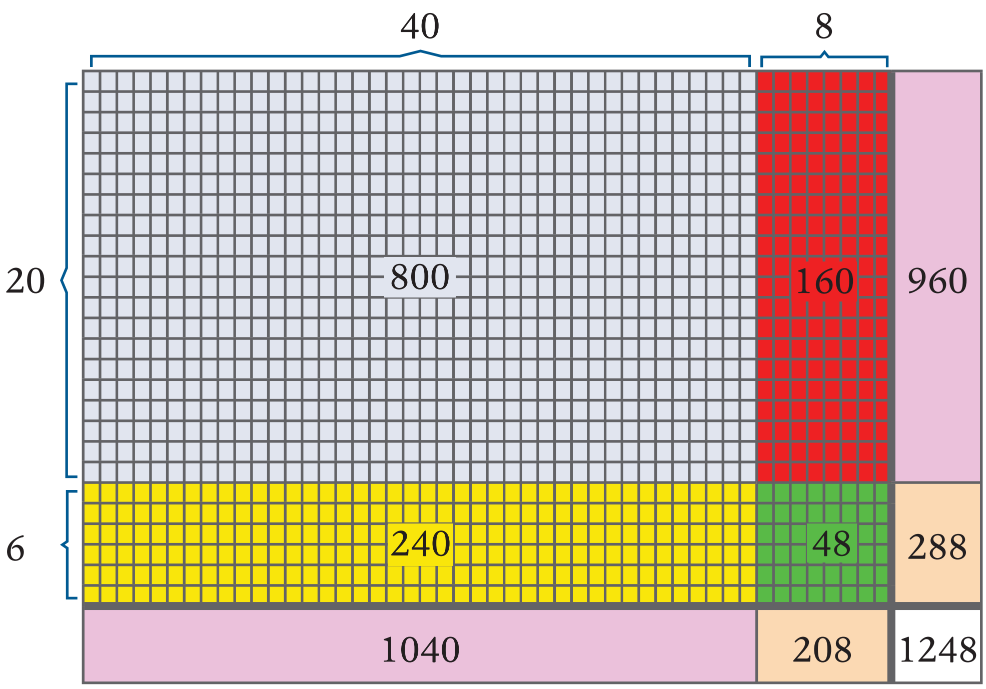
Have Student Teachers draw the intersection model for 6 x 3 (6 lines crossed by 3 lines) and again ask about the number of intersections and how that number relates to the lines that they drew. Have them experiment using the intersection model for other equations. Ask Student Teachers to explain why they think this model works.

Introduce the area model as shown earlier, in which Student Teachers draw a rectangle of 6 rows and 3 columns. Ask how this geometric model is different from the array. Have them turn their paper so that the orientation shows 6 columns and 3 rows. Remind them of the commutative property of addition and how it applies to multiplication as well. Have them draw other area models to show different equations.

Allow Student Teachers a few minutes to multiply 26 x 48, first by using the familiar algorithm and then by using another method to arrive at the answer. Most likely, Student Teachers will suggest using more 'friendly' numbers (such as 25 x 48 or 26 x 50, and then adjusting their answer). This short calculation activity is designed to prepare them for using the area model for multi-digit multiplication. (This is also an opportunity to restate that although there is one correct answer for 26 x 48, there are several strategies that can be used to find it.)

At this point introduce how the area model can be used for multi-digit multiplication. Note that this method will rely on something they did earlier in the unit: decomposing numbers.

Explain how they can combine place value with multiplication by decomposing each factor into tens and units and aligning those numbers on the top and side of the multiplication grid (as in this illustration).



After demonstrating how to multiply 26 x 48 by this partial products model, ask Student Teachers to refer to how they solved 26 x 48 by using the traditional algorithm. Do they see any similarities? Is the answer the same? Do some of the same numbers appear in both methods? Why is this so? How do the two methods relate to each other?

Distribute a Hundred Chart to each Student Teacher so that they can work with skip-counting. Have some Student Teachers colour in multiples of 2, others multiples of 3, etc. What patterns do they notice? How is this way of using a Hundred Chart to think about multiplication different from using a traditional multiplication chart? Where do they see skip-counting on the multiplication table?

X	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

End the class by posing a challenge. Ask if multiplication always makes numbers bigger, and if not, for what types of numbers this does not hold true.

Assignment

Have Student Teachers read this blog about multiplication being more than repeated addition. Ask them to consider how they might teach multiplication in such a way that will avoid having children assume that multiplication always results in a product greater than the multiplicand and multiplier.

➢ http://tinyurl.com/Mult-Repeat-Add

Unit 1/week 2, session 3: Division of whole numbers

What do Student Teachers need to know?

Multiplication and division are inverse operations. They undo the action of each other in the same way that addition 'undoes' the action of subtraction (and vice versa).

Because multiplication and division are inverse operations, there are 'fact families' of numbers in relationship with each other (such as 3, 6, and 18) that give rise to the following four equations:

$$3 \times 6 = 18$$
$$6 \times 3 = 18$$
$$18 \div 3 = 6$$
$$18 \div 6 = 3$$

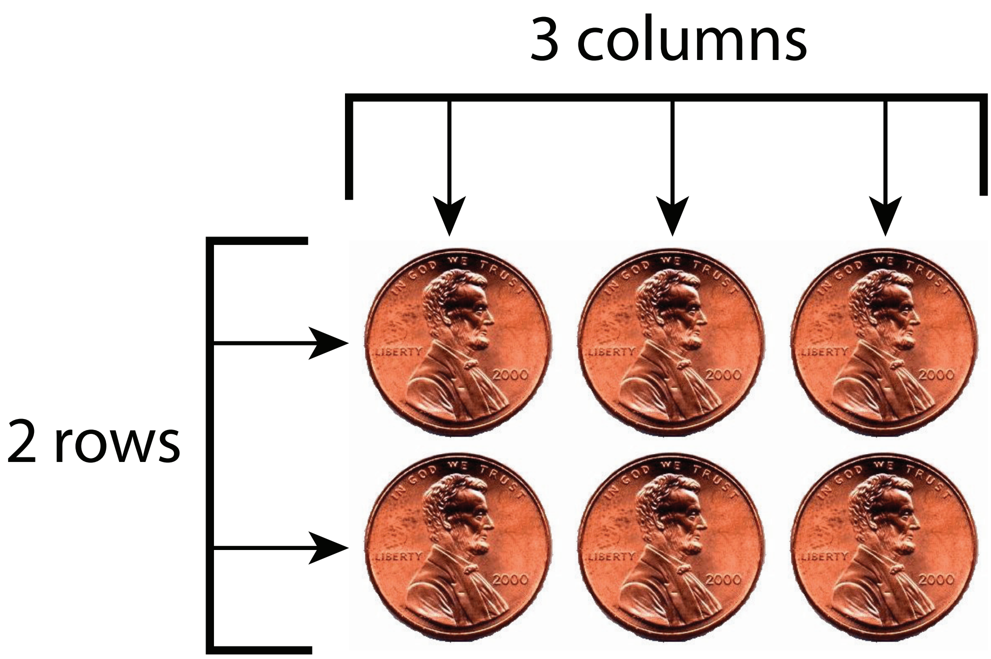
This relationship between multiplication and division parallels the relationship for addition and subtraction (as in the fact family of 5, 7, and 12 discussed last week):

$$5 + 7 = 12$$
$$7 + 5 = 12$$
$$12 - 5 = 7$$
$$12 - 7 = 5$$

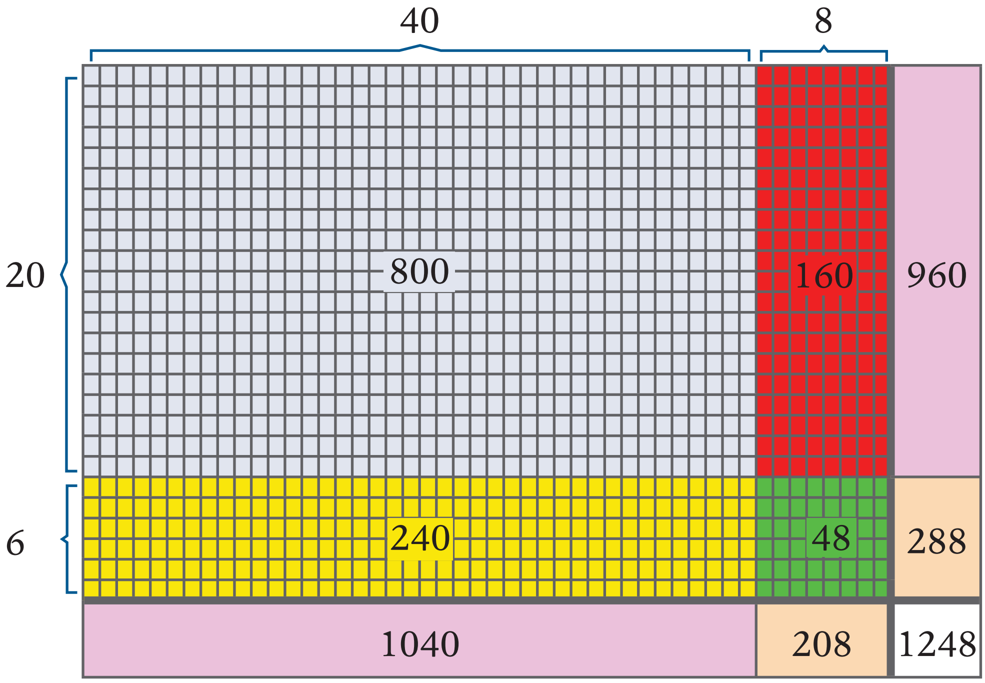
Multiplication (like addition) is commutative (6 x 3 = 3 x 6). However, division, like subtraction, is not commutative (18 ÷ 3 ≠ 3 ÷ 18).

Models for division. The array and area models that were used to visualize multiplication also can be used to visualize division. When using these two models for division, however, one needs to notice the whole (the product) first and then notice the factors (the labels for the rows and columns).

When multiplying, this array of 6 coins is a product of factors: 3 (columns) x 2 (rows). But this same array also can be interpreted as 6 coins (the total number of coins) ÷ 3 (columns) = 2 (rows).



The same is true for the following area diagram that was used to demonstrate multi-digit multiplication: 1248 (the total in the white box) ÷ 48 (on the horizontal) = 26 (on the vertical).



When multiplying using the number-line model, you begin at 0 and then make equal-sized jumps ahead. Thus, starting at 0 and making 6 jumps of 3 will land you on 18. The reverse would be true in this situation: starting at 18 and jumping back in groups of 3 six times will land you on 0.

NOTE: However, if you were to use the number line to divide 19, you would make the same 6 jumps of 3, but in this case you would land on 1. This indicates two things: 1) 6 and 3 are not factors of 19 and 2) there is a remainder that needs to be considered.

There is an additional model for division: sharing or distribution. For young children, introducing this model could be as simple as a story about sharing 10 sweets among 5 children. Each child would receive 2. The sharing/distribution model also can raise the issue of remainders; for example, what happens when sharing 11 sweets among 5 children?

Division as factoring versus division as sharing. The sharing/distribution model is less of a visual model (such as in the array and area models) and more of a dynamic model. Children can act out story problems (such as 11 sweets for 5 children) and equations ($11 \div 5 = ?$) to help them understand how a quantity can be divided among a group. Ensure that Student Teachers notice the difference between dividing a quantity into parts versus whole number factoring.

Nature of the remainder. The sharing/distribution model, with its ability to generate remainders, allows children to make real-life connections for division and consider how to express the remainder.

For example, suppose you had 11 sweets to share among 5 children. When acting this out, children will probably interpret the remainder of 1 as meaning each child will receive 2 sweets and there will be 1 left over. Whereas there may be a discussion about what to do with the leftover sweet (e.g. give it to the teacher), no young child will consider it sensible to unwrap a small sweet, try to cut it into 5 equal-sized pieces, and give 1/5 or 0.2 of the remaining sweet to each of the 5 children! In this context, the remainder is simply 1.

However, in other contexts (and in abstract mathematical terms), the answer to 11 ÷ 5 = ? as either a fraction (2 1/5) or decimal (2.2) would be both sensible and appropriate.

Symbolic representation for multiplication and division. Whereas addition has the plus symbol (+) and subtraction has the minus symbol (–), multiplication and division have multiple symbolic representations. The expression 6 times the quantity 3 could be written as 6 x 3, 6*3 (especially on a calculator), and in algebra, 6(3).

Similarly, 18 divided by 3 could be written as 18 ÷ 3, 18/3 (often found on a calculator), $\frac{18}{3}$ (with a horizontal fraction bar), or by using the long division box: $3\overline{)18}$

How do children think about these concepts?

After having used the array and area models for multiplication, children should notice a connection between the whole (the product) and its factors. Building on what they already know, connecting these models to division is the next step.

However, confusion may occur when symbolic notation related to multiplication and division is introduced. How does 6 x 3 = 18 relate to 18 ÷ 6 = 3? This is why it is important to build on two concepts children learned earlier:

- Fact families. Just as (5, 7, 12) related addition to subtraction, fact families such as (6, 3, 18) now relate multiplication to division.
- Inverse operations. Just as addition and subtraction were understood as inverse operations (undoing each other), so too are multiplication and division.

For the sharing/distribution model, it is important to give children problems to act out; for example: 'One for me, one for you, and one for you. Another for me, another for you, and another for you'. If there are any left over, note how the children try to interpret the meaning of the remainder.

In addition to providing objects (such as pencils or hard candy sweets) that can't be subdivided, have children work with items (or pictures of items) such as biscuits that could be cut into parts.

Children in the primary grades will not have the fractional or decimal vocabulary to describe the result of 11 ÷ 5 = ?, but they should be able to say that for 11 biscuits shared among 5 children, each child will receive 2 whole cookies and 'a little bit more from the leftover one'.

When addressing symbolic representation for multiplication and division, teachers need to be aware of the confusion children encounter because of the different ways that multiplication and division equations are written. Adults move seamlessly among these representations in their personal lives. However, children need to see one representation at a time, and teachers need to be precise when a new one is introduced.

When a teacher is working at the board after having taught children about the division sign (18 ÷ 3), he or she may unconsciously use the back slash (18/3) or the fraction bar with 18 above and 3 underneath the horizontal line (which is called a *vinculum*). Teachers need to be aware of these different representations that adults take for granted but which can be confusing to children. (Situations like this also might occur for older children when introducing symbols for multiplication and division on a calculator.)

What is essential to do with Student Teachers?

- Introduce the idea of multiplication and division as inverse operations.
- Link the array and area models for multiplication to these same models for division.
- Have Student Teachers consider how the number-line model for multiplication may or may not model division of whole numbers.
- Introduce the division model of sharing or distribution.
- Clarify the difference between the division model of sharing or distribution and that of products with whole number factors.
- Introduce different ways of interpreting the remainder.

Activities with Student Teachers

Begin by reminding Student Teachers that last week they used an addition chart to decompose the number 12 into (5 + 7), (6 + 6), etc., and how they then used that same chart to show subtraction (12 – 5 = 7, 12 – 6 = 6, etc.). Remind them that just as there is an inverse relationship between addition and subtraction, where one operation 'undoes' the other, there is an inverse relationship for the operations of division and multiplication: one operation 'undoes' the other, as in:

$$3 \times 6 = 18$$
$$6 \times 3 = 18$$
$$18 \div 3 = 6$$
$$18 \div 6 = 3$$

Briefly review the array and area models for multiplication, noting how these models could be used for division: beginning with the product, and then finding the factors of that product.

Have Student Teachers consider how the number-line model for multiplication may or may not model division of whole numbers by using the equations 18 ÷ 3 and 19 ÷ 3.

As the main class activity, introduce the model of sharing or distribution by having Student Teachers, in groups of three, use counters to solve the following equations:

$$18 \div 3 =$$
$$17 \div 3 =$$
$$19 \div 3 =$$

In the class summary, have Student Teachers note that in the case of 18 divided by 3, the distribution created equal shares. For 17, the distribution could be termed either '2 more' or '1 less' of equal shares. Nineteen allows for '1 more' than equal shares. Have Student Teachers describe what is happening with the remainder in each case.

As Student Teachers work with the sharing or distribution model, have them explain how it is different from products in the array and area models that were based on whole number factors.

Introduce different ways of interpreting the remainder by using story problems, such as:

- 'Suppose our class had 26 students and 1 Instructor. We want to visit a school 20 kilometres from here. Several of you have access to cars and have offered to drive. Each car can hold 6 people. How many cars and drivers do we need?'

  Student Teachers should note that although 27 ÷ 6 equals 4.5, it can also be thought of as 4 with a remainder of 3 (4 cars with 3 people remaining behind). However, in practical terms, there cannot be half a car. In this context, the remainder needs to increase by one more than the whole number quotient. Thus, while the purely mathematical answer to the problem is 4.5, the realistic answer is: you need 5 cars.

- 'Suppose I have 9 pencils to share among 5 students. How many pencils does each student receive?'

  The answer to 9 ÷ 5 is 1.8 or 14/5. But because pencils cannot be broken to be distributed, the answer is that each of the 5 students will receive 1 pencil. The remaining 4 pencils cannot be shared fairly.

- 'Suppose I have 6 large biscuits to share among 4 children. How many biscuits does each child receive?'

  The answer to 6 ÷ 4 could be interpreted as:

  - 1.5
  - 1½
  - 1 with a remainder of 2

  However, because we can cut the remaining 2 biscuits in half, it is sensible to say that each child would receive 1½ biscuits. In this case the remainder can be thought of as a fraction.

Ask how the contexts of these three examples imply different treatment of the remainder.

End the class by dividing the class into three groups. Ask each group to create a story problem in which:

- the remainder should go to the next whole number
- the remainder can be disregarded
- the remainder should be treated as a fraction or a decimal.

Have Student Teachers report their ideas.

Assignment

Have Student Teachers do the following.

- Look at this video about multiplication and division as inverse operations:
  - ➢ http://tinyurl.com/Relate-Mult-Divi
- Use this multiplication table to discover patterns:
  - ➢ http://tinyurl.com/Mult-Chart-144

FACULTY NOTES

Unit 1/week 3: Introduction to rational numbers

Session 1: Introduction to fractions
Session 2: Introduction to decimals
Session 3: Introduction to least common multiple and greatest common factor

Faculty preparation for the upcoming week (1–2 hours)

- View this PowerPoint presentation that covers models of fractions through operations with fractions (which will be discussed more fully in Semester 4 in the Mathematics II/Teaching Mathematics course):
  - ➢ http://tinyurl.com/Fraction-PPT
- View this website on models of fractions:
  - ➢ http://tinyurl.com/Fraction-3Models
- Review these decimal misconceptions:
  - ➢ http://tinyurl.com/Decimal-Misconc
- Download and print out as handouts for class (one per Student Teacher):
  - o Decimal squares:
    - ➢ http://tinyurl.com/Decimal-Grid
  - o Decimal number lines (provided as resources in the Course Guide):
    - ➢ http://tinyurl.com/NumLine-Dec
  - o Area model fraction strips:
    - ➢ http://tinyurl.com/Area-Fraction-Strips
- Have ready to bring to class:
  - o Rulers
  - o Strips cut from paper to make fraction strips
- Read the plans for the upcoming three sessions.

Weeklong overview

This week gives Student Teachers a very basic introduction to fractions and decimals–as numbers. Operations with fractions and decimals will be addressed in greater detail during the Semester 4 Teaching Mathematics course. Thus, it is not necessary for you to address operations with fractions and decimals this week. Just concentrate on Student Teachers' conceptual understanding of these two types of numbers and how they are different from whole numbers.

Recognize that young children find these two types of numbers strange, because many of the properties of whole numbers (such as 'multiplication makes a number bigger' or 'the more digits in a number, the greater its value') no longer apply.

Session 1 begins by defining a fraction as a number (on the number line). Student Teachers should not confuse this idea of a fraction as a number with the various visual and real-life representations (such as segments of a circle), however useful those models can be in helping young children begin to understand fractions.

In order to do this, Student Teachers will learn to distinguish between linear and area models by folding 'fraction strips'. By labelling the folds (as opposed to the segments), Student Teachers will create a linear model (like a ruler). When the strips, which are of equal length, are laid out, sequentially equivalent fractions will become evident.

Session 2 introduces decimals as a special type of fraction—a fraction whose denominator is a multiple of 10 but which is shown in a different format: 0.1 looks different from 1/10. However, it is the same number written in a different format. The value of the number remains the same. Young children find it difficult to understand this concept.

Models for illustrating decimals will include both a grid and a linear model (a number line from 0 to 1 that is divided into hundredths).

Session 3, which introduces the Greatest Common Factor (GCF) and Least Common Multiple (LCM), builds on concepts introduced last week: factors and multiples.

Unit 1/week 3, session 1: Introduction to fractions

What do Student Teachers need to know?

A fraction is a number. Just as many children (and adults) assume that addition is joining sets, subtraction is 'taking away', and multiplication is 'making things bigger', they also consider fractions as 'parts of things', such as an apple that has been cut into half or fourths. But when looking at models that clarify the real nature of a fraction as a number, the linear model (a number line, or in real life, a ruler) is perhaps the best way to make this important concept clear to children.

Models for fractions. When we considered multiplication, we looked at it from three dimensions: 1) 0 dimension, the set model with items as points in a plane, 2) one dimension, the linear model of intersecting lines, and 3) two dimensions, the area model of cells on a grid (and the multi-digit multiplication mats). These three models hold true for fractions, but there is at least one additional model to consider: three dimensions, volume.

What is the whole? One of the most important concepts related to fractions is the nature of the whole. Suppose I have one apple and cut it into three equal pieces. Each piece is 1/3 of the whole. However, if I have two apples and cut each into thirds, I now have 6 pieces. What does 1/3 look like now? One-third of the first apple and 1/3 of the second apple have become 2/6 of the whole (which is now two apples).

This leads to another question about the 'whole'. Suppose the whole (in a recipe) is 1¼ kilos of flour. What is half of that? How might I model this? Would a drawing help?

The important issue here is that students' conception of fractions for one item (the single apple) will eventually need to move to fractions of multiple items (two apples) and fractional situations (1¼ kilos).

Equivalent fractions. Consider the example of the two apples cut into thirds, where 2/6 is 1/3 of the whole; or a metric ruler, where ½ is equivalent to 5/10. Renaming a given fraction to one with a different denominator is the basis for the 'least common denominator' that allows for operations with fractions.

How might we address that 1¼ kilos of flour when we try to divide it in half and then discover we need to deal with a denominator in eighths? Or when we double the recipe and find we need 2½ kilos of flour? How did our original number with its denominator of 4 move to denominators of 8 or 2?

How do children think about these concepts?

Young children's thinking about fractions is usually quite confused. When hearing 'one-half' in adult conversations, children often assume that all fractions are one-half, or that one-half means any part of the whole. Thus, children need to learn that there are fractions other than one-half and that fractions are equal parts of a whole (a connection to the 'fair shares' model for division).

Children also apply whole-number thinking to fractions, assuming that if 6 is greater than 2, then 1/6 must be greater than ½.

Because fractions are usually introduced by the area model of a circle or rectangle, children often assume that ¼ of a circle is what ¼ means. This is why children need to see and work with multiple models (set, linear, volume) of fractions to see what ¼ looks like in different models.

Exposing children to equivalent fractions (e.g. 1/2 = 2/4 = 3/6 = 4/8 = 5/10) allows them to notice a pattern indicating how the numerator and denominator are related.

What is essential to do with Student Teachers?

- Introduce the conceptual model of fractions in various dimensions.
- Emphasize the nature of the whole.
- Explore equivalent fractions and mixed numbers.
- Compare fractions.
- Relate each of these points to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers in groups of no more than four to brainstorm for three minutes about all they know about fractions. Ask for and chart their responses.

Introduce the various dimensional models for fractions, asking Student Teachers to give real-life examples for set, linear, area, and volume.

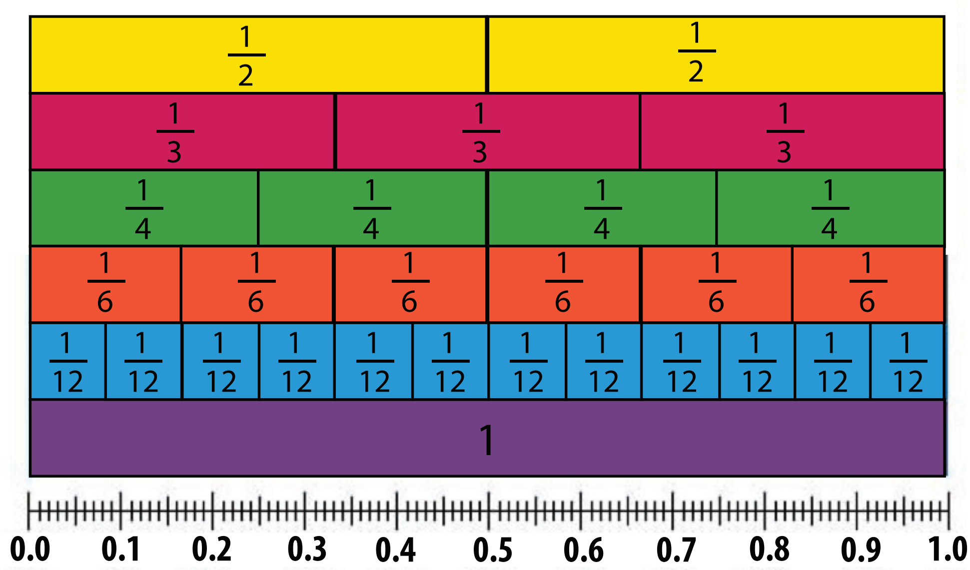
Have Student Teachers experiment with the linear model for fractions by giving them narrow strips of paper that they can fold into halves, thirds, fourths, sixths, eighths, ninths, and twelfths. Challenge them to find a way to create fifths and tenths.

Ask Student Teachers to label their fraction strips on the folds, which creates a linear model like a ruler, so that the folds on the fourths strip would be labelled 1/4, 2/4, and 3/4. (Note in the following diagram that all the fractions are labelled on the segments, which translates into an area model, but the decimal equivalents are noted where the folds would lie.)

Ask Student Teachers about where zero halves and two-halves are. Have them line up their fraction strips in order of increasing denominators to display patterns of equivalent fractions. Have them name the various equivalent fractions for ½. What about one-half on the thirds strip?

Introduce mixed numbers by having Student Teachers lay their one-half strip end to end with that of another student. What is the whole now? What is one-fourth of 2? Three-fourths of 2?

Finally, have Student Teachers compare various fractions. Which is greater: 4/10 or 4/6? 3/5 or 5/3? 5/6 or 5/8? How can you tell (without converting them to decimals)?



Assignment

Distribute this handout of an area model for fraction strips:

➢ http://tinyurl.com/Area-Fraction-Strips

Unit 1/week 3, session 2: Introduction to decimals

What do Student Teachers need to know?

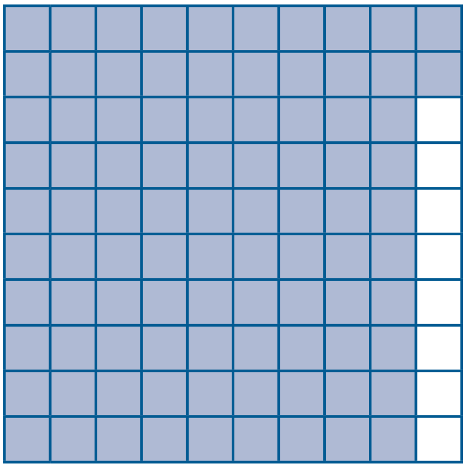
Decimals are a way to write fractions using a numeral's position in base-10 place value. When working with whole numbers, Student Teachers saw that each numeral increased by a magnitude of 10 moving to the left and decreased by a magnitude of 10 moving to the right. A number such as 23 could be written with a decimal point as 23.0 without changing its value.

Students need to develop 'decimal sense' to understand how decimals are named and the quantity they represent. One way to address this is to work with a number line that goes from 0.0 to 1.0, with points in between indicating tenths.

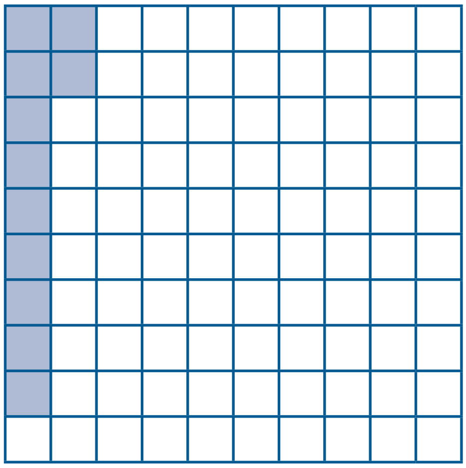
A second way is to use a 10 by 10 blank grid in which a coloured-in row or column indicates tenths. It is important that children understand how this grid is different from the Hundred Chart. In the Hundred Chart, each cell is worth 1 and the whole is worth 100.

In the case of the decimal grid, each grid is only one unit, and each cell is 1/100 of the whole. Thus, two decimal grids would be worth 2.0, but two small coloured-in squares would be worth 0.02.

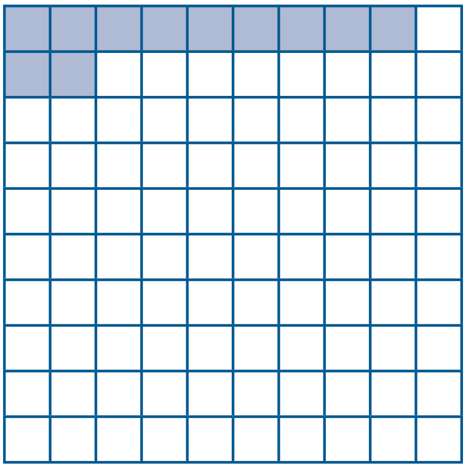
Model 1



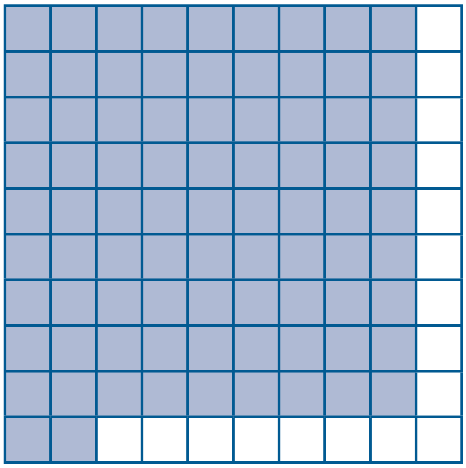
Model 2



Model 3



Model 4



Comparing and ordering decimals. Placing a group of decimal numbers in order is a challenging task for children. Consider: 6.6, 6.7, 6.08, 6.55, and 6.551. All five of these numbers have a six in the units place. But some are shorter numbers with only two digits, while some are longer numbers with up to four digits. All of which gives rise to confusion (see below regarding how children think of these).

Fraction-decimal equivalents. Decimals, as mentioned previously, are used as an alternative to the a/b model for writing fractions. As such, every fraction has its decimal equivalent, which is either truncated after several places (such as 1/8's equivalence of 0.125) or repeating (such as 1/3's equivalence of 0.33333 …), which can be indicated by a short line over the *repetend*, or repeating portion.

How do children think about these concepts?

Most of children's misconceptions about decimals are rooted in whole-number thinking. For example, being familiar with the magnitude of natural numbers, they assume that a number with many digits must be a large number. Extending this to decimals, the child mistakenly believes that .283 is greater than 2.5 because the first number has more digits, thinking that 'longer' decimals have a greater value than 'shorter' ones.

Other children may become focused on the denominator and think that any number of tenths must be greater than any number of hundredths (or thousandths) because tenths are greater than hundredths (or thousandths). These children believe, for example, that 6.45 is greater than 6.731 because thousandths are very small parts of a number.

There is also confusion about how decimals relate to fractions. For example, some children believe that 1/8 and 0.8 are the same quantity—often because they learned that 1/10 and 0.1 are equivalent.

What is essential to do with Student Teachers?

- Emphasize that decimals are a way to represent fractions whose denominators are multiples of 10 and is an extension of place value, and as such every fraction has a decimal equivalent.
- Compare and order decimals with a number line and a decimal grid in order to better understand the quantity a decimal fraction represents.
- Relate each of these points to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers in groups of no more than four to brainstorm for three minutes about what they know or recall about decimals. Chart their responses.

Ask Student Teachers what a *decimal point* means and help them articulate extending numbers to the right of the decimal point as being a pattern consistent with what they already know about place value for whole numbers.

Distribute the decimal grid and discuss how it is only one unit but subdivided into hundredths. Ask them to show one-tenth, then one-hundredth, and then one-thousandth. (For thousandths, they will have to divide one of the cells into 10 parts.) Then have them find seven-tenths and ask how they would write it in decimal format. What would seventy-three hundredths look like? How would they write it?

Have Student Teachers begin to compare decimals by using the grids. Which is greater, 0.45 or 0.54? 0.08 or 0.8? How do they know?

Continue using the decimal grid, having Student Teachers describe how they would find 7/50 or 1/4. How do they approach the task? How do they know their answer is correct?

Distribute the 0.0 to 1.0 decimal number-line handout and have Student Teachers label the tenths in decimal notation on one of the number lines, making sure they put a 0 to the left of the decimal point. After doing this, have them find various hundredths and label those.

Then have the Student Teachers label one of the number lines with any fractional equivalents they know. Ask about 1/8 and 1/6. How would they know where to write those numbers?

Give Student Teachers a series of numbers (such as those listed earlier) and ask them to put them in order from least to greatest. Discuss the various types of erroneous thinking that are common in children, and then ask how they think children might put this group of decimals in order.

Assignment

Have Student Teachers read this article on children's misconceptions about decimals:

- http://tinyurl.com/Decimal-Misconc

Have Student Teachers create a list of five decimal numbers that could highlight children's confusion about decimals and write them on five slips of paper. Have them ask a child to put the decimals in order while describing his or her rationale for the choices. This is a type of performance assessment called a *clinical interview*. Have the Student Teachers take notes on what the child says and does.

Unit 1/week 3, session 3: Greatest Common Factor, Least Common Multiple, introduction to operations with fractions

What do Student Teachers need to know?

Factors and multiples were mentioned in the class session on multiplication and division. These concepts become especially important as children begin to add, subtract, multiply, and divide fractions.

To find the Greatest Common Factor (GCF) of two numbers, for example 16 and 24, we would list all the factors for each (for 16: 1, 2, 8, 4; for 24: 1, 2, 12, 3, 8, 4, 6) and then look for the greatest one that is common to both lists. In this case it would be 8.

The GCF is most often used when renaming fractions. For example, for 5/20, the factors of 5 are 1 and 5, and the factors of 20 are 1, 2, 4, 5, 10, and 20. In this case, the GCF is 5. Both the numerator and denominator can be divided by 5, creating the equivalent fraction ¼.

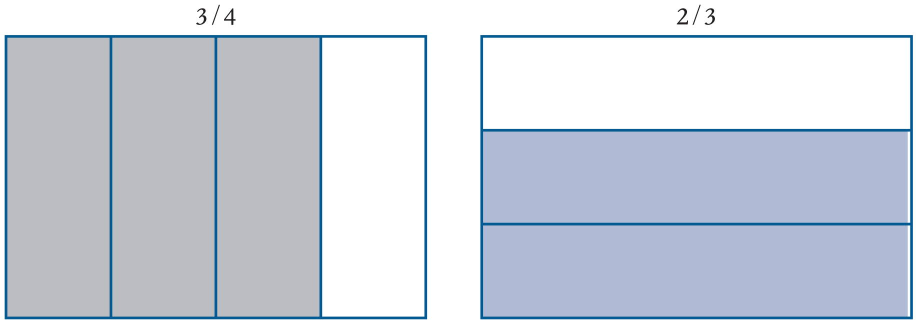
To find the Least Common Multiple (LCM) for 16 and 24, we would generate multiples for each: (16, 32, 48, 64) and (24, 48, 72, 96). In this case the LCM would be 48. Another way to find the LCM is to break each number into its prime factors—(16 = 2 x 2 x 2) and (24 = 2 x 3 x 4)—and then cast out the duplicates. With a 2 cast out from each set of factors, we are left with the remaining factors 2, 2, 3, and 4, which, when multiplied, give 48.

In primary school mathematics, the LCM is often referred to as the Lowest Common Denominator when adding fractions, such as 5/8 and 3/4. In this case the LCM is 8, which allows for renaming ¾ to its equivalent 6/8, with a resulting sum of 11/8. The same would hold true for subtraction: the renamed 3/4 (now 6/8) – 5/8 = 1/8.

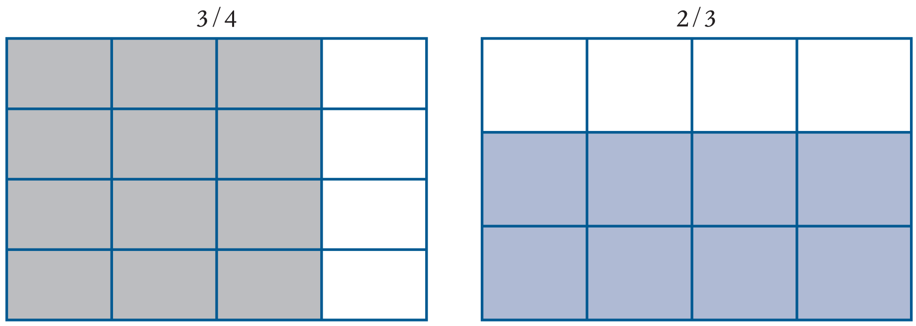
Operations with fractions. As mentioned above, when adding and subtracting fractions, it is necessary to discover the LCM so that each fraction has the same denominator. This is also where children's familiarity of equivalent fractions (as with fraction strips laid out in rows) is important so that they begin to have an intuitive sense of common equivalents.

When multiplying fractions, children need to make sense of the fact that multiplication will not always 'make things bigger'. For example, in the case of 3/4 x 2/3, the answer is 1/2, which is less than either of the original numbers.

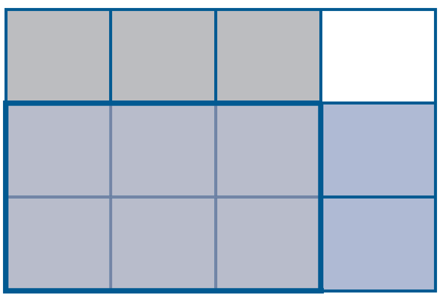
To help children understand this, a visual model can help. The following two rectangles are shaded to show 3/4 and 2/3.



Now, partition each rectangle as shown below. Note that they now have the same number of cells (12). (This is, in fact, a way to represent the Lowest Common Denominator. We can also use this method to compare fractions, noting here that 3/4 fills 9 cells, whereas 2/3 only fills 8. Thus, 3/4 is greater than 2/3.)

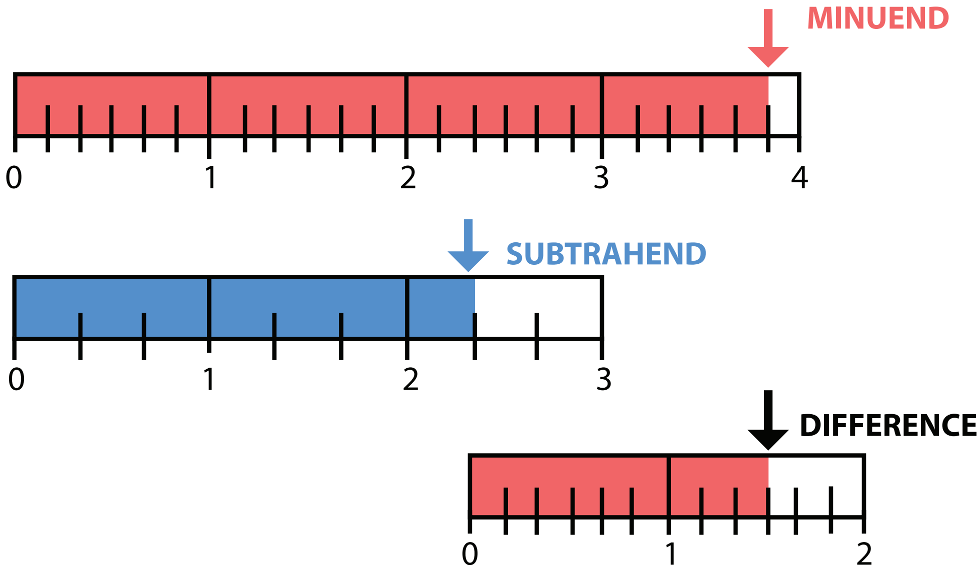


If we overlap the rectangles, the overlapping portion is 3/4 of the 2/3. Count the overlapping squares. The overlapping portion is 6/12, or its equivalent, 1/2.



How do children think about these concepts?

When working with fractions, children commonly look at an addition or subtraction example and simply add (or subtract) the numerators, then add (or subtract) the denominators. Thus, for 5/8 + 3/4 it would not be uncommon for a child to come up with a mistaken answer of 8/12. In subtraction, this same type of thinking might result in 3/4 – 5/8 = 2/4, with the child reversing the minuend and the subtrahend.



$3\frac{5}{6} - 2\frac{1}{3} = 1\frac{1}{2}$

Minuend **Subtrahend** **Difference is correct!**

As mentioned in the session on multiplication, because of whole-number thinking, children often assume that multiplication always 'makes things bigger' than its factors.

When multiplying fractions (unlike when adding and subtracting them), you operate across the numerators and across the denominators, multiplying each.

What is essential to do with Student Teachers?

- Present the concept of the GCF and LCM, how to find them, and how they are used when working with fractions.
- Address how operations with fractions are both similar to and different from operations with whole numbers.
- Provide Student Teachers with visual models and strategies to help them conceptualize operations with fractions.
- Relate each of the above to children's thinking.

Activities with Student Teachers

Have Student Teachers add 1/4 and 1/3 using any method they choose. Have them share their methods and record on chart paper.

Introduce the Greatest Common Factor and ask Student Teachers to find the GCF of various number pairs. Ask why they think this topic is relevant to this week's work on fractions.

Introduce the Least Common Multiple and ask Student Teachers to find the LCM of various number pairs. Again, ask about the relevance of this topic.

Ask if, when they added 1/4 and 1/3, they were intuitively using the GCF and LCM. Ask how the LCM might be related to denominators.

Note how the addition and subtraction of fractions follow many of the rules for natural number addition and subtraction—as long as there is an LCM for the denominator.

Move to the multiplication of fractions by introducing the grid model, which helps explain why fractions of fractions are less than either of the two original numbers. Pose the problem 3/4 x 2/3. Ask Student Teachers to estimate the answer. Then use the rectangular model to ask why multiplication of these two fractions that are less than one resulted in an answer less than each of the original fractions.

After focusing on the operation of multiplication, ask how this same visual model can be used to show the Least Common Denominator and equivalent fractions.

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 1/week 4: Per cents, ratios, rates

Session 1: Per cent and percentages
Session 2: Ratios
Session 3: Rates

Faculty preparation for the upcoming week (1–2 hours)

- Read Chapter 4 from the free e-book *Mathematical Education of Teachers*, available at:
  - ➢ http://tinyurl.com/Math-Educ-of-Teachers
- Review the article 'Interpreting Fractions, Units, and Unitizing', which discusses part-to-part and part-to-whole ratios:
  - ➢ http://tinyurl.com/Fractions-Units
- Review this website, which gives a detailed description for using a grid to understand percentages:
  - ➢ http://tinyurl.com/Illum-Percent-Grid
- Download and print (two per Student Teacher):
  - o Per cent grid:
    - ➢ http://tinyurl.com/Illum-Percent-Grid-Handout
- Bring to class:
  - o Rulers
  - o Graph paper
- Read the plans for the upcoming three sessions.

Weeklong overview

Session 1 this week will extend last week's work with fractions and decimals to address the concept of per cent and how to calculate percentages.

Session 2 will introduce the concept of ratio and proportion, emphasizing the difference between part-to-part and part-to-whole. This session will also question Student Teachers' understanding of cross-multiplying and ask them to use their knowledge of equivalent fractions as an alternative strategy for solving ratio problems.

Session 3 will introduce rates, which will be addressed more fully in the algebra unit.

Unit 1/week 4, session 1: Per cents and percentages

What do Student Teachers need to know?

Last week's emphasis on fractions and decimals laid the groundwork for understanding per cents and percentages. Students should consider a particular decimal fraction such as 0.5 and ask how that representation translates into a fraction in a/b format (1/2) and then into a per cent (50%).

Models for per cents. A decimal grid can be thought of as a per cent grid, where, instead of calling a shaded area '75 hundredths' or '0.75', it can be thought of as 75%.

As children move from fractions to decimals to per cents, they need to consider benchmarks such as division by 10 being translated into multiplication by 1/10 or 0.1. Later they will relate this to '10 per cent of' a number.

This idea of a benchmark fraction of 1/10 allows children to work from the basic 1/10 in order to consider either half of a tenth or double one-tenth. Knowing that half of 0.1 is equal to 0.05 allows the student to calculate the sum of 0.1 + 0.05 as 0.15 and then translate that into 15 per cent. Doubling 10 per cent means 20 per cent, tripling gives 30 per cent, etc.

The procedure for changing decimals to per cents involves 'moving the decimal point two places to the right and then adding the per cent sign'. For example, 0.02 would become 2% (not 20%), 0.25 would become 25%, and 0.125 (equivalent to the fraction 1/8) would become, by moving the decimal point two spaces to the right, 12.5% (not 125%, which children sometimes write by simply placing the per cent sign after the decimal's last digit).

However, simply moving the decimal point is a procedure. Even if children can do it correctly, they may not understand why it works. They need to realize that *per cent* literally means parts 'per hundred'. Thus, to move from a decimal (which is based on 1.0) to a per cent (which is based on 100), we need to multiply the decimal by 100:

$$\begin{array}{r} 0.75 \\ \underline{\text{x } 100} \\ 75.00 \end{array}$$

Similarly, to go from a per cent to a decimal, we need to divide the per cent by 100.

Per cents in the real world often address prices and statistical data. When dealing with prices, there are two common scenarios: discounts and taxes.

In the case of a discounted price, the per cent of the price is subtracted from 100 per cent (the original price). Thus, for a $100 purchase with a discount of 20 per cent, I would end up paying only 80 per cent of the original price. To compute this discount, I would subtract the per cent of the discount from 100 per cent and use the remainder as the multiplier. ($100 x 80%, which I would calculate as $100 x 0.8 or $100 x 8/10.) If I need to pay sales tax on my purchase, however, I would add that per cent to the 100 per cent of the original cost. Thus, if a sales tax rate is 8%, I would need to add 8% to 100% and then multiply my original $100 purchase by 108% (1.08) to determine my final cost.

How do children think about these concepts?

Children's transition from decimals to per cents rests on their familiarity with decimal notation and their ability to think about 'moving the decimal point'. For example, suppose that a child understands that 0.60 is somewhat more than 1/2. As he or she transitions to the percentage model for that quantity (60%), it is helpful to create a classroom chart of three columns labelled 'fraction', 'decimal', and 'per cent'.

Having a visual that shows that 6/10, 0.60, and 60% are simply different names for the same quantity can help children connect these three different representations that all describe the same number. Once again, this relates to the concept of equivalent fractions and fractions to decimals to per cents. This can also be connected to earlier work done with whole numbers, in which a single quantity '12' could be represented by various expressions, such as 5 + 7, 3 + 9, 13 – 1, etc.

Children often fail to consider the whole when working with per cents. For example, they may think that 50 per cent means that there are 50 objects in the whole. Or they may think that 50 per cent of one year is 5 months, when it is actually 6 months, again failing to recognize the whole.

As with fraction-to-decimal conversions, in which children mistakenly assume that 1/6 is 0.60, they also may think that when working with fraction-to-per cent conversions that 1/6 is 60 per cent.

When using a calculator to work with per cents, the input needs to be the per cent's decimal equivalent. The need to know the per cent's decimal equivalent is another reason why it is important to become fluent in moving between fractions, decimals, and per cents, because even if children don't use calculators in the classroom, they may use them at home, and certainly they will need to know how to use them in the future.

What is essential to do with Student Teachers?

- Connect how three representations (fraction, decimal, and per cent) describe the same quantity.
- Discuss how to calculate percentages using the discount and tax models. Explain how this relates to the multiplication of fractions and decimals (80 per cent of a given price will be less than the original, whereas 108 per cent of the item's original cost will be more than the original).
- Relate each of these points to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers in groups of three or four to brainstorm for several minutes about what they know about percentages. Chart their responses.

Refer to the decimal grids used last week and ask how their work with them might be rephrased in terms of percentages. Note any comments Student Teachers make that relate per cents to fractions and decimals and how they describe these relationships.

Briefly remind Student Teachers about the work they did with multiplication of fractions and ask how this will relate to working with per cents. If necessary, make clear the connection among all three representations of the same quantity.

To offer an opportunity to work with per cents, present Student Teachers with this problem, which they should solve in pairs or small groups:

'You find a 20,000 PKR dress on sale at a discount of '60% off'. What does the dress cost?'

This is an occasion of using adult experiences to explore mathematics. Later these Student Teachers will need to create age-appropriate scenarios that will engage children in thinking about how mathematics relates to their real-life experiences.

Have Student Teachers create a similar problem with a discount that has a context for children in grades 6 or 7.

Assignment

Have Student Teachers study this website, which gives a detailed description for using a grid to understand percentages:

- http://tinyurl.com/Illum-Percent-Grid

Unit 1/week 4, session 2: Ratios

What do Student Teachers need to know?

Proportional thinking, like multiplicative thinking, develops over time with its roots in students' understanding of fractions, decimals, and per cents. Ratios and proportions are used to compare two quantities to answer questions such as 'What is the ratio of men to women in our class?', in which we are comparing two parts of the whole to each other (part-to-part).

This is different from the type of comparison made when asking the question 'What proportion of our class has a laptop computer?', in which the comparison is part-to-whole. Note that these types of part-to-whole questions could be rephrased as 'What per cent (or fraction) of our class has a laptop?'

Models for ratios and proportions. Ratios and proportions can be numeric, or they can be geometric. For a numeric example, consider the ratio of lemonade concentrate to water. Each can of concentrate has directions printed on the label that says it should be diluted with three cans of water. Thus, the resulting mix of 1 can of concentrate with 3 cans of water gives a total of 4 cans' worth of juice.

This mixture can be thought of in several different ways. If we consider the relationship part-to-part, the mixture would have a ratio of 1:3 (1 part concentrate to 3 parts water).

We can also think of the mixture as part-to-whole, where the concentrate is 1/4 of the mixture (1:4) and the water is 3/4 of the mixture (3:4). Either relationship, part-to-part or part-to-whole, is valid, but we need to be clear about which type of relationship we are discussing.

How do children think about these concepts?

Children often are not aware that the order of terms in a ratio is important. For example, if I have four children and only one is a girl, the ratio of girls to boys is 1:3, whereas the ratio of boys to girls is 3:1.

Children also are confused by part-to-part versus part-to-whole ratios that refer to the same situation. For example, using this scenario, I have three times as many sons as daughters (part-to-part), but 3/4 of my children are boys, where the 4 represents the total number of children (the whole) in my family.

As mentioned, although children can be taught cross-multiplying as a quick way to solve proportions, they usually have no idea why this works. Even though the written description of using equivalent fractions might seem involved, it actually makes the mathematics of solving proportions more sensible to children.

What is essential to do with Student Teachers?

- Compare and contrast the part-to-part and part-to-whole models for thinking about ratios and proportions, referring back to the nature of the whole that was discussed when studying fractions and per cents.
- Have Student Teachers devise ratio problems that relate to real-life situations.
- Introduce solving proportions for an unknown by using equivalent fractions.
- Relate each of the above to children's thinking.

Activities with Student Teachers

Begin by describing the scenario of a family with 3 boys and 1 girl. Have Student Teachers quickly write all the ways they can think of to express that relationship using both words (e.g. 'I have three times as many sons as daughters' or 'three-fourths of my children are boys') and in symbolic representation (3:1, 1/4, etc.).

Ask for and chart their responses. Be aware that some Student Teachers will use the comparison model of subtraction to discuss the relationship ('I have two more sons than I have daughters').

Use Student Teachers' responses to launch a comprehensive class discussion about 1) how part-to-part and part-to-whole relationships are different, 2) part-to-whole relationships can be expresses as fractions or per cents, and 3) the order of numbers in a ratio is important.

As part of this discussion, have Student Teachers generate several scenarios in which ratios may be found in real-life situations.

Ask Student Teachers how they could solve $8/36 = 20/x$. If they mention cross-multiplying, ask them to explain how it works. Then ask how they might use what they learned about equivalent fractions to solve for x. Have them work in partners and use this method to discover a solution.

Assignment

Have Student Teachers read 'Interpreting Fractions, Units, and Unitizing', with its explanation of part-to-part and part-to-whole thinking:

➢ http://tinyurl.com/Fractions-Units

What do they think about the last section and how it relates to adding fractions? Can they think of a real-life scenario where adding part-to-part might occur?

Unit 1/week 4, session 3: Introduction to rates

What do Student Teachers need to know?

Constant rates of change are ratios that show a relationship between an independent and dependent variable. In real-life situations, this might be the number of calories per cup of rice or the number of kilometres (km) per hour.

For example, if I drive at a constant speed (the independent variable), I can calculate the distance I drove for various lengths of time (the dependent variable).

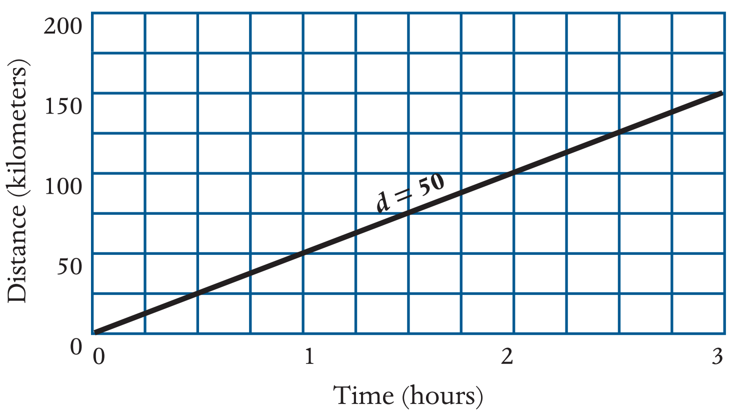
Note that the word *per* separates the two units in the rate.

Multiple representations. When working with rates, it is important for students to see various ways change can be represented: in a table, graph, and equation. In the kilometre per hour scenario, a table would show the following distances:

Rate: 50 km per hour

t: Time in hours (x)	d: Distance in kilometres (y)
0.5	25
1.0	50
1.5	75
2.0	100
2.5	125
3.0	150
5.0	250
n	$50n$

The graph of this scenario would look like this:



Notice that the graph shows a straight line, indicating a constant rate of change, and the distance travelled for any particular hour (or fraction of an hour) on the x-axis can be found by the line's location relative to the y-axis.

The third representation would be an equation, in this case $d = 50t$, where d is the distance and t is the time. Note that if the time and distance were known, we could use simple algebra to discover the rate: $d/t = 50$.

Understanding constant rates of change will become important later in algebra, when they will be linked to linear equations and slope.

Rate problems in the real world might deal with prices (rate per unit), interest rates (rate per time period), or currency conversion rates. In fact, conversion tables (Celsius to Fahrenheit, for example) are based on rates.

How do children think about these concepts?

Often children are presented with only the equation model when asked to work with rates. They are then asked to perform calculations, usually in the context of a word problem such as 'How many kilometres will you drive in 2 hours if your average speed is 50 kilometres per hour?' However, the very next problem assigned may ask them to 'calculate how many times your heart beats in an hour based on 110 heartbeats per minute'. These two problems, focused only on one numeric answer, do not illustrate, as a table does, that a pattern of change emerges for rate problems and the variables actually vary.

Without their seeing that ratio problems can be graphed on a coordinate grid, children will have difficulty understanding both the place of rates in linear relationships and how slope is related to a constant rate of change when they begin to study algebra.

Recall the scenario in the previous session of the family with 3 sons and 1 daughter, where the ratio of boys to girls was 3:1, but the ratio of girls to boys was 1:3. Order mattered. The order of the units for rate matters, too. There is a major difference between finding the distance (km per hour) versus finding the time it takes to go one kilometre (hours per km).

Conversion rates are more complex than simpler rates such as km per hour. Thus, even if children understand what the rate means, they will need excellent computational skills to make the conversion. This is where in real life using even a simple calculator makes the conversion quick and easy.

What is essential to do with Student Teachers?

- Explain how constant rates of change are related to ratios and proportions.
- Have Student Teachers construct and interpret tables and graphs to solve a rate problem.
- Have Student Teachers create an equation to express the rate.
- In a whole group summary discussion, modelling how a similar discussion would take place in the classroom, compare and contrast the three different representations.
- Relate each of the above to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers to share the rates they use in their daily life. As each rate is shared, note the use of the word *per* to name the two units involved. Also show how rates can be written with a '/' sign, as in 'km/hour'. Ask how this representation relates to division. For each example that has been shared, ask the Student Teachers what a reasonable rate might be (such as 3 km/hour walking, 30 km/hour driving). Do they know any specific rates, such as converting centimetres to inches or vice versa?

Note the rates that they shared are called 'constant rates of change' and that they will explore several ways to represent them. Mention the terms *independent variables* and *dependent variables* and ask Student Teachers why these are accurate descriptions for the units in the rate.

Introduce a rate problem such as the one above, having Student Teachers work in pairs, one using the rate of 30 km/hour, the other with a rate of 40 km/hour. First they will create a table beginning with various numbers for x, find y, and then determining the value of y for any x. Ensure they know the traditional format for setting up this two-column chart, with the rate labelling the top.

Next, ask them to create a graph using the data from their table. Ensure that Student Teachers know what a Quadrant I graph is and how to scale it appropriately to fit the problem's numbers.

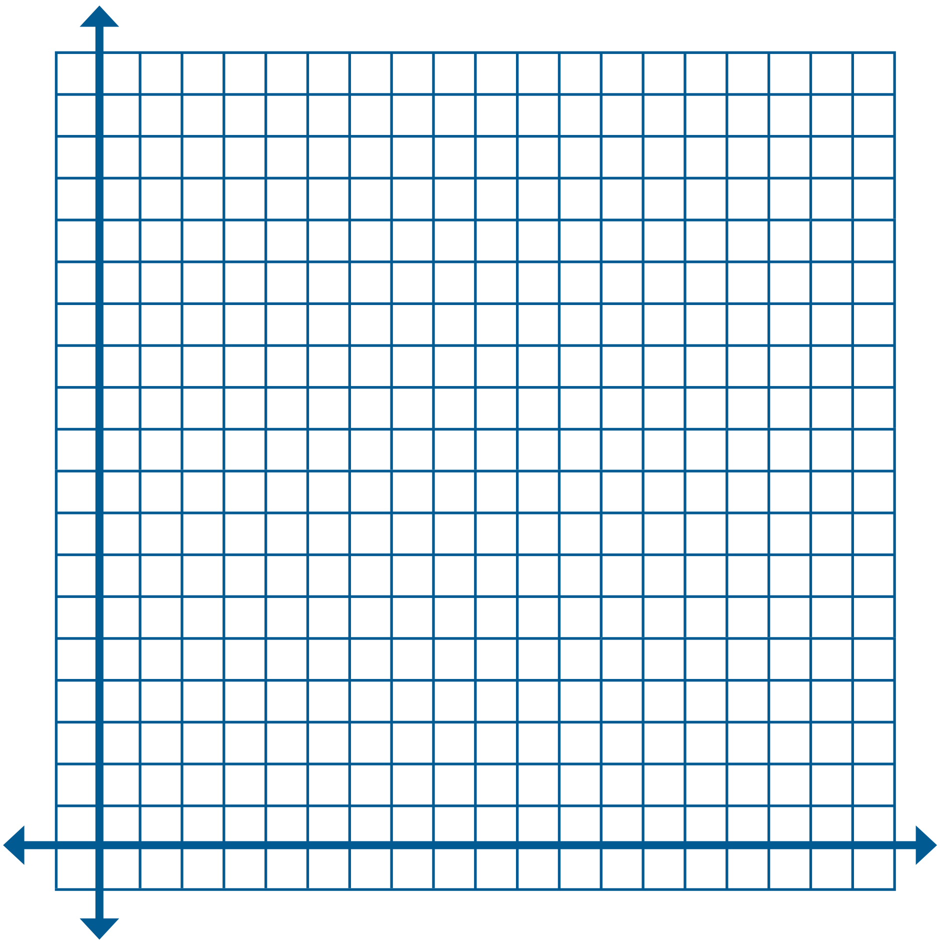


Figure: Quadrant I graph

Have each pair of Student Teachers compare their two graphs and discuss with each other:

- How are they alike?
- How are they different?
- What is the 'equation of the line' on each of the two different graphs?

End the session by having a whole-class discussion that summarizes their answers to the above questions and asks additional questions, such as:

- How could you find the distance for times between your last numerical entry and n by using the table?
- How would distances beyond your last numerical entry be shown on the graph?
- How are the table, graph, and equation related to each other?
- Predict what your graph would look like if you were walking at 3 km/hour. Quickly add that data to your graph. Was your prediction correct? What two things does this graph compare?
- How does the difference between the y-values in your table relate to the rate?
- Show the table with the rate of 50 km/hour that includes fractional values for x and missing entries between 3.0 and 5.0. Ask what they notice about your table that might be different from theirs.
- Ask if there is a consistent difference between the y-values in their tables. If so, how does this pattern relate to the rate? If not, why not?

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 1/week 5: Integers, integer operations, reflection on mathematical processes

Session 1: Introduction to integers
Session 2: Addition and subtraction of integers
Session 3: Multiplication and division of integers, reflection on maths processes

Faculty preparation for the upcoming week (1–2 hours)

- Look through the following websites that address integers and operations with integers:
  - ➢ http://tinyurl.com/ThinkMath-Integers
  - ➢ http://tinyurl.com/Integers-Interactive
  - ➢ http://tinyurl.com/Integer-Runner
  - Addition:
    - ➢ http://tinyurl.com/Integer-Add-Applet
  - Subtraction:
    - ➢ http://tinyurl.com/Integer-Subtract-Applet
- Do the maths:
  - Practice adding integers with two-colour counters using this interactive applet:
    - ➢ http://tinyurl.com/Integer-Chip-Trading
  - Practice subtracting with two-colour counters using 'Subtraction with Integer Chips: Teacher Notes', which you will also use as a class handout in Session 2.
  - Practice integer addition and subtraction with the interactive 'Number Line Runner':
    - ➢ http://tinyurl.com/Integer-Runner
- Print out the following handouts for class:
  - Integer number line (one page, two per Student Teacher):
    - ➢ http://tinyurl.com/Integers-NumLine
  - 'Subtraction with Integer Chips: Teacher Notes' (available as a resource with the Course Guide)
  - 'Subtraction with Integer Chips: Student Worksheet' (available as a resource in the Course Guide)
  - 'Add and Subtract Integers Fact Sheet' (one page, one per Student Teacher):
  - http://tinyurl.com/Integer-Reference
  - Multiplication and Division of Integers Reference Sheet:
    - ➢ http://tinyurl.com/Integer-Mult-Div-Reference
  - 'End-of-Unit Reflection' worksheet (one per Student Teacher; available as a resource in the Course Guide)
  - Walk the line script (one page, for Instructor reference):
    - ➢ http://tinyurl.com/Walk-Line-Script

- Prepare and have ready to bring to class:
    - Large white beans and markers (or crayons) for Student Teachers to create two-colour counters
    - A six-metre strip of paper folded into twelfths, to be used as a walk-on integer number line
    - Graph paper
    - Sticky notes of two different colours
- Read the plan for the upcoming three sessions.

Weeklong overview

Session 1 will introduce a new type of number: integers. Several models for integers will be shown: a number line that extends 'to the left' beyond zero and two-coloured counters. (Although two-coloured counters can be purchased, it is easy and less expensive to use large white beans, which Student Teachers can colour using felt-tip markers or crayons.)

Session 2 will address addition and subtraction with integers using the number line model and the two-coloured counter model.

Session 3 will be devoted to multiplication and division of integers using the model of arithmetic patterns.

Because this is the last session of the 'Numbers and Operations' unit, time needs to be allotted for Student Teachers to reflect on the mathematical process standards that were used during this unit:

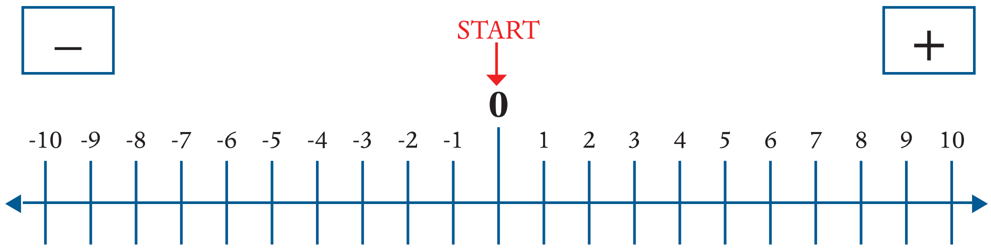
- Modelling and multiple representations
- Mathematical communication
- Problem-solving
- Connections: both to real-life situations and to other areas of mathematics (algebra, geometry, and information handling)

Unit 1/week 5, session 1: Introduction to integers

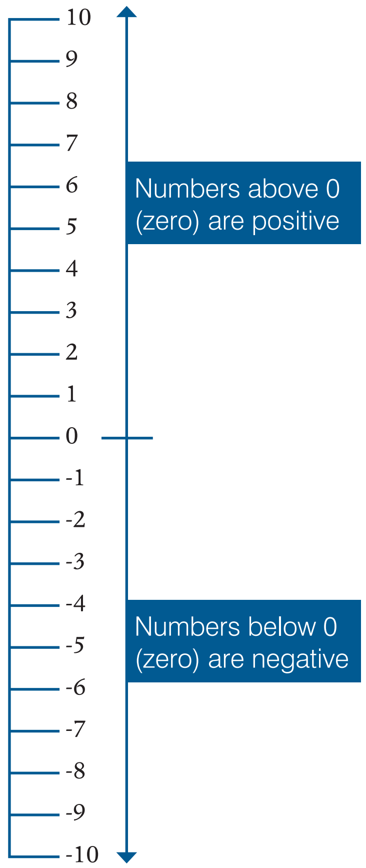
What do Student Teachers need to know?

Integers, like fractions, are another type of number in our number system. Models for integers include the following.

The number line. Up to this point, Student Teachers have worked with number lines that began at 0 and went on indefinitely in a positive direction. They also created and labelled fraction strips to show numbers on the number line between whole numbers (especially those between 0 and 1). Now Student Teachers will extend the number line beyond zero to address negative integers.



We can also think of a vertical number line.



Notice that simply by introducing the integer number line, notation for the negative sign also has been introduced. Although published materials rarely make this distinction, it is helpful when writing on the board or creating worksheets to use a short dash, or hyphen (-) or a superscript dash (¯) in front of the number to indicate the negative sign, and a longer dash (–) to signify the operation of subtraction, such as in the expression 2 – (-3), which reads 'two minus negative three'.

Student Teachers will use integer number lines such as those above not only to experiment with addition and subtraction of integers in the next session but also to explore several other topics today: opposites, absolute value, a four-quadrant coordinate plane, and the use of integers in real-life situations.

Two-colour counters. This is a model that will be used in the next two sessions on operations with integers. If you have not worked with two-colour counters, it is crucial that you practice with them before introducing and using them in the next two sessions.

Commercial two-colour counters usually have black or red on one side and white or yellow on the other. However, two-colour counters can be made by simply using a bag of large white beans and marking one side of a bean with a crayon or felt-tip marker. Detailed instructions on using two-colour counters to model addition and subtraction with integers are included in the 'Integer Chips: Teacher Notes' handout that will be used in the next two sessions.

Numerical patterns. Although not a hands-on way of working with integers, number patterns are a way to stress that our number system is an orderly and logical one.

$$4 - 1 = 3$$
$$4 - 2 = 2$$
$$4 - 3 = 1$$
$$4 - 4 = 0$$
$$4 - 5 = ?$$
$$4 - 6 = ?$$

For each positive integer there is a negative integer called its opposite. For example, the opposite of 2 is -2; the opposite of -5 is 5. In a pair of opposites, each number is equidistant from 0 on opposite sides of the number line. More important, when two opposites are added, their sum equals zero. This concept of 'zero-sum pairs' is the foundation of using two-colour counters to model integer addition and subtraction in the next session.

A Positive Integer | A Negative Integer | Zero Pairs $1 + -1 = 0$ | Zero Pairs $2 + -2 = 0$

It is also important that Student Teachers recognize that zero is neither positive nor negative.

Finally, although this section has covered integers, eventually Student Teachers will need to realize that in order for our number system to be consistent, there will be numbers (fractions such as -½, decimals such as -0.3, and later irrational numbers) that lie between negative integers.

How do children think about these concepts?

When children begin to work with integers, they need to understand multiple concepts and integrate them. This is because they are being exposed to a new kind of number in our number system, a type of number that differs from whole or natural numbers.

Recall that understanding and integrating concepts happened when children first learned about fractions. For children to develop 'fraction sense', they needed to connect new terminology (*fourths* rather than *four*), a new symbol for notation (the fraction bar), a new relationship between two numbers (a/b), and several new visual models to help them understand this new concept. And all of this needed to be done before they could meaningfully compute with fractions, decimals, and per cents.

Similarly, this same long list of tasks relates to integers as children begin to develop 'integer sense'. This integration is a complex procedure that means, just as with fractions, children will need both multiple models for integers as well as time and relevant activities to make sense of this new type of number.

Just as children often use whole-number thinking when considering the number of digits in decimals (thinking incorrectly 3.0001 must be greater than 3.1), they tend to apply whole-number thinking to integers. A common misconception is that a number such as -14 must be greater than 3 because -14 has more digits. Children need to work extensively with number lines to realize that any number to the right is greater than any number to its left, and that any positive number (even 1) is always greater in value than any negative number (even -100).

Giving children rules for integer operations before they understand integer concepts is almost always ineffectual. Not only do children not comprehend the meaning of what they are doing, but the various rules simply become a list to memorize—and often forget. If Student Teachers continually need to refer to a reference sheet when performing operations with integers, this is a sign that they have not yet internalized basic concepts about integers.

As mentioned above, children often become confused by the subtle distinction in notation of the hyphen (indicating a negative integer) and the minus sign (indicating the operation of subtraction). As such, Student Teachers need reminders to call -3 'negative 3', not 'minus 3'.

Even very young children can be introduced to negative numbers simply by adding an extension to the left of their classroom number line. (Some commercial number lines for the primary grades actually have the negative numbers from -1 to -10 written in red.)

What is essential to do with Student Teachers?

- Introduce integers as a new type of number with a new type of notation.
- Introduce three integer models: number line, two-colour counters (and zero-sum pairs), and mathematical patterns.
- Clarify vocabulary associated with integers, especially negative versus minus, and zero-sum pairs.
- Relate each of these points to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers to share what they know and remember about integers. Chart their responses. New ideas can be added over the next two sessions when they address operations involving integers.

Distribute one copy of the integer number line to each Student Teacher, asking how this number line is different from the one they used before. Using the number line as a tool, ask about number pairs such as 2 and -2. What do they notice about them? As Student Teachers respond, build their ideas into a discussion of opposites. If no one mentions it, ask about the sum of 2 and -2 to lead into a discussion of zero-sum pairs.

Briefly introduce the idea of zero-sum pairs and ask how they relate to the concept of opposites. (To demonstrate zero-sum pairs on the board, use sticky notes of two different colours.)

A Positive Integer | A Negative Integer | Zero Pairs $1 + -1 = 0$ | Zero Pairs $2 + -2 = 0$

For the last activity, have Student Teachers use graph paper to create a four-quadrant coordinate grid with both x and y parameters as -10 to 10. Have Student Teachers think of the grid as the intersection of the x-axis (the horizontal integer number line) and the y-axis (the vertical integer number line). Have them plot these points: (5, 4) (-5, 4), (-5, -4), and (5, -4). What do they notice?

Briefly show the model of pattern continuation as described earlier and ask for a rationale of why it is a valid way to think of negative numbers.

Assignment

To prepare for the next class, have Student Teachers work with the following two-colour chip applet for the addition of integers. Ask them to consider how the applet models zero-sum pairs.

➢ http://tinyurl.com/Integer-Chip-Applet

Unit 1/week 5, session 2: Operations with integers

What do Student Teachers need to know?

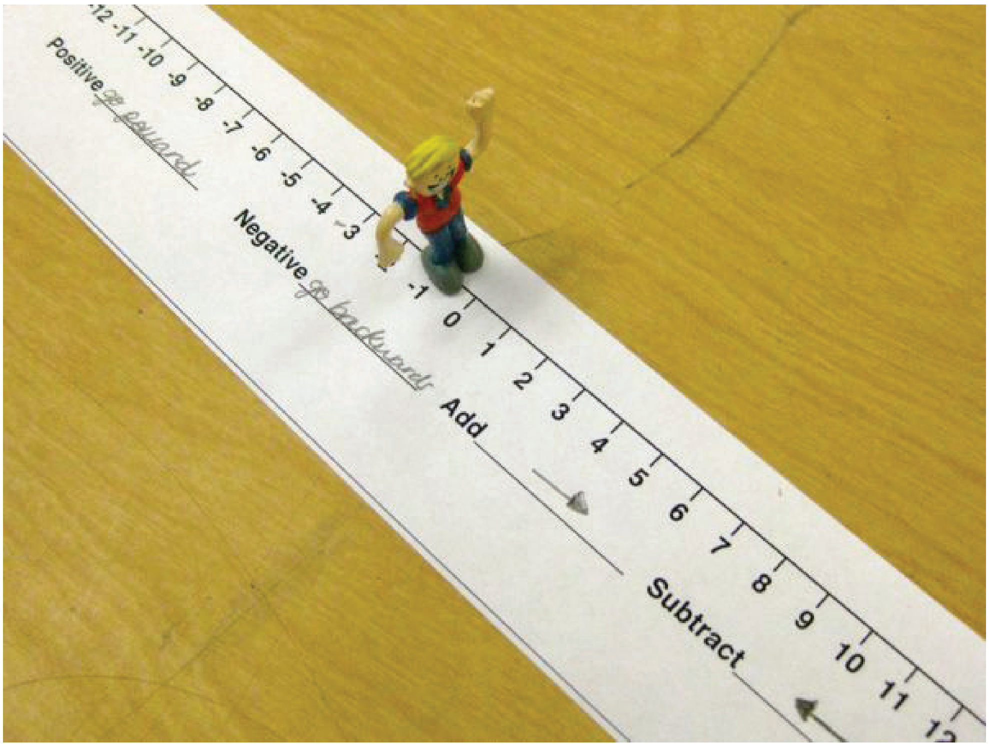
The goal for this lesson is to have Student Teachers develop algorithms for operations with integers by engaging in various activities. This is in contrast to having Student Teachers simply memorize the rules, versus them understanding underlying concepts so well that they can reconstruct the rules when necessary.

Operations are actions. In the models and activities described below, there is a distinction between the negative sign of a number and the minus sign indicating the operation of subtraction. Thus, Student Teachers will physically move objects or act out the operations to build their understanding. These would include actually walking on a number line in prescribed ways to model addition and subtraction with integers, and moving two-colour chips into zero-sum pairs.

Different operations are best served by different models.

When using (and actually walking on) a number line to model an equation, there are two directionalities that we need to consider:

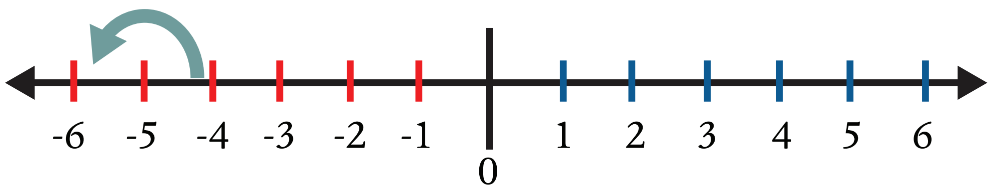
1) The sign for the number, the direction to which we face: positive (to the right) or negative (to the left).
2) The sign for the operation, an action, in which we 'walk' forward for addition or backward for subtraction.



We will be using variations of one basic equation to explore the concept of adding and subtracting integers with the goal of having Student Teachers see patterns and come to generalizations.

For example, to model the equation (-4) + (-2) = x, I would start by standing at -4, the first number in the equation, looking toward the left of 0, in the negative direction. Then I consider the second term. Because it is -2, I would stay turned in that negative direction.

Once I have the signs of my numbers in place, I need to consider the operation. Because the operation is addition (+), I walk forward and land on -6.



To model the equation (-4) + (+2) = x, I would begin on -4, looking in the negative direction. Then because 2 is positive, I would turn to face the positive direction, toward the right of 0. Because the operation is addition, I would walk forward and land on -2.

On the other hand, if the operation was subtraction, (-4) – (+2), I would begin at -4, looking in the negative direction, turn to model positive 2, and then walk backward 2 steps and land on -6.

Finally, if the equation were (-4) – (-2) = x, I would start at -4 facing to the left. Then I would stay turned to the negative direction because of -2. However, because the operation says subtract, I would walk backward 2 steps and land on -2.

As Student Teachers 'walk the talk' and record their results, they will begin to realize that there is a pattern, a relationship among the four equations:

(-4) + (+2) = -2
(-4) – (+2) = -6
(-4) + (-2) = -6
(-4) – (-2) = -2

This exercise should also give rise to the generalization about rules for adding and subtracting integers as well as the realization that subtracting a number is the same as adding its opposite.

This is another example of 'fact families', in which, for example, the whole numbers 3, 4, and 7 could be organized into the following four equations:

$$3 + 4 = 7$$
$$4 + 3 = 7$$
$$7 - 3 = 4$$
$$7 - 4 = 3$$

You might also show this applet (also listed in the assignments at the end of this class) to help clarify this 'turn and walk' idea:

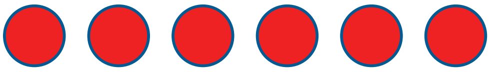
- http://tinyurl.com/Integer-Runner

Recall from Week 1 that two models for the subtraction of whole numbers were 'finding the difference' and 'taking away'. Although an equation such as -4 – (-2) can be modelled by either the number line or two-colour chips, the number line is a more useful model for problems that have a context that requires finding a difference (such as temperature), while the two-colour chips model may be more effective in situations such as 'taking away' (such as handling sums of money).

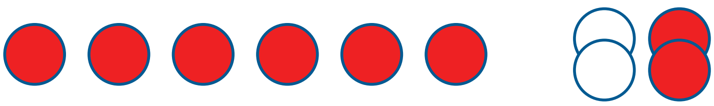
(A full tutorial, 'Integer Chips: Teacher Notes', about how to use two-colour chips, is included in the material for the first hour of the week. Please be sure to work through all the problems yourself before introducing this model in class.)

Briefly, the idea is that when we add opposites such as 2 and -2, the sum is zero. Hence we have the term 'zero-sum pairs'. And zero, as we know, can be added to any number without changing the value of the original number.

Consider the example $-6 - (+2) = x$, in which negative numbers are represented by red chips and positive numbers are represented by white chips. I set up -6:



However, because there are no white (positive) chips to take away, I need to add enough zero-sum pairs to allow me to subtract a positive two.



When I take away the two white (positive) chips, the remainder is -8.

How do children think about these concepts?

- All the concerns cited in the previous session, 'Introduction to integers', are of concern when children add and subtract integers.
- When using two-colour counters, Student Teachers need to understand that adding zero-sum pairs does not change the value of a number.

What is essential to do with Student Teachers?

- Develop algorithms for the operations of addition and subtraction with integers by engaging in meaningful activities that foster in-depth understanding of these operations.
- Relate this point to children's thinking.

Activities with Student Teachers

Begin by asking Student Teachers to recall a) what they remember about operations with integers and b) how they learned this topic when in school.

Let Student Teachers know that they will be experimenting with integer activities that may be new to them, that they will need to be patient with themselves as learners, and that they may experience the same feelings of uncertainty that children experience when working with these activities for the first time.

Have several Student Teachers create a walk-on integer number line using the six-metre strip of paper you brought to class.

Introduce the idea of directionality—that we can use the number line to model facing in a negative or positive direction, and then walk either forward to show addition or backward to show subtraction.

Using the 'Walk the Line' script have several Student Teachers walk the line to model addition and subtraction of integers. Have several Student Teachers walk the line in response to integer equations such as those described earlier. Record the equations and their answers on the board so that Student Teachers can consider a pattern that might develop into an algorithm.

Distribute a copy of the integer number line and work through several addition problems and subtraction problems, using arrows showing directionality to solve each problem.

Distribute the beans and have Student Teachers colour them with crayons or markers to create two-colour counters. Also distribute the tutorial 'Integer Chips: Teacher Notes'.

Review the idea of zero-sum pairs and how to use this concept to solve the equations on the tutorial sheet. Work through the tutorial slowly while Student Teachers do actual hands-on work with the beans, modelling each scenario.

When Student Teachers have completed the tutorial, distribute the packet 'Integer Chips: Student Worksheet'. Have Student Teachers work in pairs or small groups to solve the problems and write their own 'subtraction rules' after comparing and contrasting various types of equations.

At the end of class, have Student Teachers report on patterns they have discovered (with both the number line and the two-colour counters) that led to their developing rules for the addition and subtraction of integers.

Assignment

Distribute the 'Add and Subtract Integers Fact Sheet', which organizes the rules for addition and subtraction with integers:

➢ http://tinyurl.com/Integer-Reference

To help Student Teachers understand the number-line model for addition and subtraction of integers, have them look at this website of a 'virtual runner':

➢ http://tinyurl.com/Integer-Runner

To help Student Teachers understand the two-colour chip model for addition and subtraction of integers, have them work with these interactive virtual manipulatives:

- Addition:
  ➢ http://tinyurl.com/Integer-Add-Applet
- Subtraction:
  ➢ http://tinyurl.com/Integer-Subtract-Applet

Unit 1/week 5, session 3: Multiplication and division of integers, reflection on practice

In this last session of this unit, Student Teachers consider the mathematical processes used during the unit:

- Modelling and multiple representations
- Mathematical communication
- Problem-solving
- Connections both to real-life situations and to other areas of mathematics (algebra, geometry, and information handling)

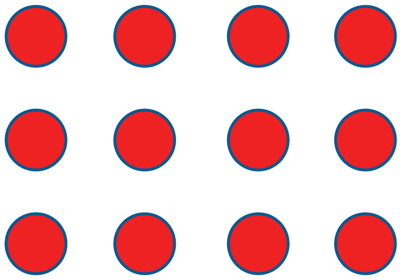
Please allow time after the mathematics content of this session for Student Teachers to complete the reflection pages and have a whole-group discussion about their thoughts.

What do Student Teachers need to know?

Although physical and visual models were helpful when learning about the addition and subtraction of integers, they are less helpful when addressing integer multiplication and division. To introduce integer multiplication and division, have Student Teachers build on what they already have learned about addition and subtraction of integers, multiplication and division of whole numbers, pattern continuation, and the overall consistency of our number system. Thus, this session will be somewhat more abstract than the prior session.

Models for multiplication of integers. Although we discussed earlier that multiplication was more than repeated addition, the repeated addition model is helpful in understanding integer multiplication. This may be shown by:

- Repeated leaps of the same size on a number line.
- Laying out 3 rows of -4 (red) chips:



- Repeated addition equations, such as: 3 x (-4) = (-4) + (-4) + (-4) = -12

Notice that the sign of the product is negative, as it would be if we started at 0 and took 3 leaps of (-4) in the negative direction.

Because multiplication is commutative, the expression (-3) x 4 can be rewritten as 4 x (-3) and then solved in the same manner as above. Again, the sign of the product is negative.

After doing several of these, it should be obvious one can simply multiply the two numbers, and if one is positive and the other negative, the sign of the product is negative.

It is also important for Student Teachers to understand that when multiplying a negative number by a positive number, the result is never greater than the factors. Have them look at the results of their multiplication and see if they notice this pattern before mentioning it.

However, when two negative numbers are multiplied, the result is positive. In this situation, the number line and repeated addition are less useful models to help understand why this is so.

One model that is helpful is to look at patterns with the understanding that our number system is logical and consistent.

$$(-3) \times 4 = -12$$
$$(-3) \times 3 = -9$$
$$(-3) \times 2 = -6$$
$$(-3) \times 1 = -3$$
$$(-3) \times 0 = 0$$
$$(-3) \times (-1) = ?$$

The answer cannot be -3; that was the answer to (-3) x 1. So it must be something else. To maintain the pattern of the answer becoming 3 greater each time, (-3) x (-1) must be +3.

Recall that in Session 1 of this week, we discussed the concept of opposites. Because (-1) is the opposite of 1, the answer to (-3) x (-1) must be the opposite of (-3) x 1.

However logical this is, Student Teachers may not be convinced. Here is where a real-life example can make this concept easier to understand. One useful negative context is time past. For example, I took a loan from the bank to buy a car. Every month Rs10,000 is taken out of my bank to pay back the loan. This can be thought of as -Rs10,000. How much more was in my account three months ago (-3 for time past)? The answer is that three months ago I had +Rs30,000 more in my account.

Models for division of integers. Because division is the inverse of multiplication, the same models described above apply. Although repeated addition is useful in explaining integer multiplication, Student Teachers probably will find it more difficult to apply that idea to the division of integers. Thus, beginning by building on what they learned about patterns and the idea of the inverse may be more helpful. Note that by creating the chart in this format, the idea of dividing a negative by a negative immediately results in four examples of a positive product.

(-3) x (+4) = -12	so -12 ÷ (-3) = +4
(-3) x (+3) = -9	so -9 ÷ (-3) = +3
(-3) x (+2) = -6	so -6 ÷ (-3) = +2
(-3) x (+1) = -3	so - 3 ÷ (-3) = +1
(-3) x 0 = 0	so 0 ÷ (-3) = 0
(-3) x (-1) = +3	so +3 ÷ (-3) = -1

Fact families. Just as we could build a fact family for the inverse operations of multiplication and division of whole numbers:

3 x 4 = 12
4 x 3 = 12
12 ÷ 4 = 3
12 ÷ 3 = 4

We also can create fact families for integers:

(-3) x +4 = -12
(-3) x (-4) = +12
-12 ÷ (-3) = +4
12 ÷ (-3) = -4

How do children think about these concepts?

All the concerns noted in this week's Session 1 still apply to the multiplication and division of integers.

Children become confused by the apparent contradiction that when you add two negative numbers the result is negative, but when you multiply two negative numbers the result is positive.

Even if children can apply the rules for operations with signed numbers, it is no guarantee they understand what those rules mean.

What is essential to do with Student Teachers?

- Introduce models for multiplication of integers, beginning with visuals such as the number line and two-colour counters, then moving to patterns.
- Introduce models for division of integers, continuing the emphasis on patterns and the consistency of our number system.
- Relate each of these points to children's thinking.
- Have Student Teachers reflect in writing and then in a whole-class discussion on their learning during Unit 1.

Activities with Student Teachers

Begin by asking what Student Teachers recall about multiplication and division of integers and by what methods they learned them.

Introduce integer multiplication by reminding Student Teachers how they could model negative numbers with two-colour counters. Ask them how they might multiply 4 x (-3) by using the two-colour counters. How might they model the same 4 x (-3) by using a number line?

Remind Student Teachers that although multiplication is more than repeated addition, repeated addition can help them understand integer multiplication. Ask them how they would multiply 4 x (-3) by using the repeated addition method. Then ask how we could multiply (-3) by 4, noting the commutative property of multiplication if no one suggests it.

As you begin to introduce the multiplication of two negative numbers, ask Student Teachers how they might model this. Note where there may be misconceptions. Also note anyone who suggests using patterns and ask for elaboration. If necessary, recreate the pattern list above on the board and ask what is happening to the products. Emphasize that our number system is consistent and that because of opposites, the product of (-3) x (-1) cannot be equal to the product of (-3) x (+1).

Ask if anyone can share a real-life example of multiplication of two negative integers. If not, offer the idea of past time (if modelled on a time line); past time can be considered negative if today is considered point 0. Present the car loan example to help Student Teachers understand the concept.

Introduce the division of integers by building on the patterns developed for multiplication earlier.

Mention inverse operations and how the operations of multiplication and division of whole numbers are related. Ask if Student Teachers can create a multiplication and division fact family for the whole numbers 3, 4, and 7. Then have them create a fact family for integers.

Distribute the reflection sheet and give Student Teachers about five minutes to fill it out.

Finally, have a whole-group discussion of how the different mathematical processes helped them better understand the concepts they will need to teach.

Assignment

Have Student Teachers download the following fact sheet that organizes the rules for multiplication and division with integers:

- http://tinyurl.com/Integer-Mult-Div-Reference

UNIT 2

ALGEBRA

FACULTY NOTES

Unit 2/week 1: Patterns, algebra as generalized arithmetic, children's algebraic thinking

Session 1: Patterns as fundamental to understanding algebra
Session 2: Algebra as generalized arithmetic
Session 3: The algebraic thinking of young children

Faculty preparation for the upcoming week

- Read the following articles:
  - 'The Algebra of Little Kids: Language, Mathematics, and Habits of Mind':
    - http://tinyurl.com/ThinkMath-Early-Algebra
  - 'Algebra in the Elementary Grades? Absolutely!':
    - http://tinyurl.com/M-Burns-Articles
- Look through the following websites that address patterns and algebraic thinking:
  - Patterns in numbers and shapes:
    - http://tinyurl.com/Patterns-Numbers-Shapes
  - Exploring patterns:
    - http://tinyurl.com/Exploring-Patterns
  - Powerful patterns:
    - http://tinyurl.com/Powerful-Patterns
  - Patterns in Pascal's triangle:
    - http://tinyurl.com/Patterns-Pascal-1
  - Difference of squares:
    - http://tinyurl.com/Diff-of-Squares
- Download and print out for Student Teacher use:
  - Pascal's triangle worksheet:
    - http://tinyurl.com/Patterns-Pascal-2
  - Centimetre grid paper:
    - http://tinyurl.com/GridCm
  - 'What Do Students Struggle with When First Introduced to Algebra Symbols?' PDF version available at:
    - http://tinyurl.com/Struggle-Algebra
  - 'Reflection on "What Do Students Struggle with When First Introduced to Algebra Symbols?"' (one copy per four-person group; available in Course Guide resources)
  - 'Patterns in Numbers and Shapes' (available in Course Guide resources)
- Bring to class:
  - Crayons, coloured pencils, or markers
  - Various small objects that can be arranged into patterns
- Read through the plans for this week's three sessions.

Weeklong overview

In Session 1, Student Teachers will come to understand the importance of patterns as fundamental to algebraic thinking and why exploring patterns is introduced to children in the early grades. Children first begin work with repeating patterns. They need to notice a pattern and determine the 'pattern unit', such as AB AB or ABB ABB. Then they are asked to duplicate the pattern and finally to extend it. Note that children are also exposed to patterns in songs, stories, and physical activities. Later children are introduced to 'growing' patterns of both pictures and numbers as described in the document handout 'Patterns in Numbers and Shapes' (in the course guide resources).

Young children should also look for number patterns in tables and charts, such as in this addition table of sums. How many 2s and 12s are there inside the grid? Compare that to sums of 11 and 3. What about the number of 7s? What would the table look like if we coloured in sums that were only represented once in one colour, twice in another colour, etc.? What pattern would emerge?

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Session 2 is devoted to the relationship between arithmetic and algebra, with algebra being a form of generalized arithmetic. This session is unlike most of the prior sessions. It will include far more discussion and fewer maths activities.

Student Teachers will need to read the article 'What Do Students Struggle with When First Introduced to Algebra Symbols?', which can either be read on its website for homework or printed out by the Instructor for distribution at the end of Session 1.

Session 3 will address young children's algebraic thinking in a more activity-centred mode.

Unit 2/week 1, session 1: Patterns and algebraic thinking

What do Student Teachers need to know?

Patterns are found in nature, in numbers, and in everyday products and activities. They may be as concrete as an AB AB row of coloured blocks or as intangible as the repeating days of the week.

There are repeating patterns and growing patterns.

Patterns can be duplicated and extended.

Although patterns are emphasized in the 'Algebra' unit, they were also part of the 'Numbers and Operations' unit.

It is especially important for children to look for and discover number patterns.

How do children think about these concepts?

Children begin by noticing that there is a pattern and then take the next step, identifying the repeating unit. (In the upper grades, children may be asked if a pattern exists in a given dataset.)

If given a pattern of coloured cubes arranged AB AB AB, children easily may be able to duplicate the pattern on grid paper using the same colours. However, if they are asked to duplicate the AB AB pattern by using two different types of coins, they may become confused, as the attributes of colour and shape are different. They do not yet see the abstraction of AB as the repeating pattern unit.

When asked to extend a block pattern (by adding more coloured blocks to the row), young children usually are able to do this once they have determined the repeating unit.

However, they fail to realize that the pattern can be extended forever—if only there were an infinite number of blocks. They tend to assume that the pattern ends when the materials run out.

Teachers need to point out less obvious types of patterns such as the surface design of fabric, the daily schedule (which is a pattern of routine), the repeated refrain of songs, etc.

Growth patterns are less obvious to children. Once again, the child needs to be able to notice and duplicate the pattern. When the child is young, he or she is usually working with a visual arrangement of shapes.

The challenge comes when the child is asked to extend the pattern, which is a more sophisticated task. After working with visual growth patterns, it is important for children to see numerical growth patterns. This can be as basic as the multiples of 2 in skip-counting.

What is essential to do with Student Teachers?

- Identify patterns in the immediate environment.
- Describe, represent, and extend repeating patterns.
- Introduce the concept of growing patterns and have Student Teachers describe and represent them both graphically and numerically, and then extend them.
- Introduce the idea of a 'pattern rule' that sets the foundation for functions in more formal algebra later on.
- Work with Pascal's triangle to discover patterns of numerical relationships.

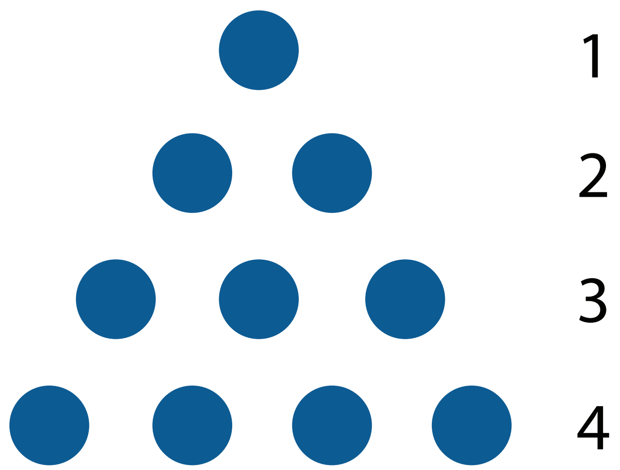
Activities with Student Teachers

Begin by having Student Teachers identify visual patterns in the classroom environment. Then ask about other types of patterns that they perceive in their everyday lives. Prompt them to go beyond visual patterns to numerical ones and intangible ones. Notice if all the examples given are repeating patterns. If anyone suggests a growth pattern, ask them to explain this idea and how it differs from repeating patterns. (If no one gives an example of a growth pattern, simply reserve discussion until you introduce the concept later in the session.)

Distribute the centimetre grid paper. Using small objects such as coins or cubes, create a pattern that Student Teachers can translate into coloured squares on their grid paper. Ask about extending the pattern. What did they need to perceive in order to do this? The grid paper is only several centimetres wide. Does the pattern stop at the end of the grid paper?

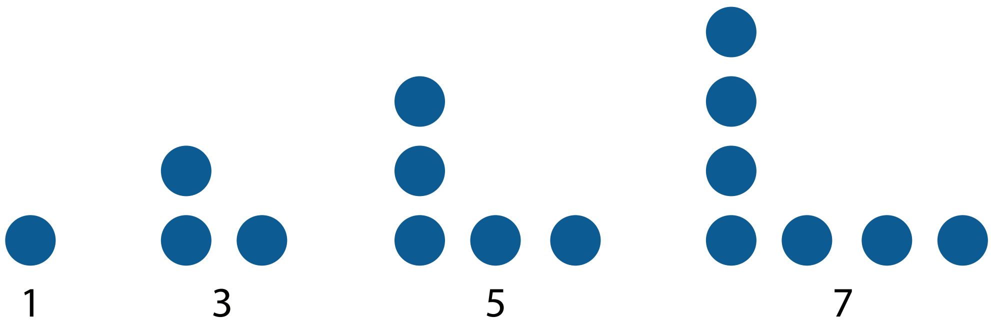
Ask what other kind of repeating patterns there are (e.g. the repeating decimal of 1/3, the repetition of television shows from week to week, the months of the year, etc.).

Introduce the concept of growth patterns by drawing a simple pattern, such as:



Have Student Teachers duplicate and extend the pattern on their grid paper.

Then ask them to consider numerical growth patterns, something as simple as counting by 2s starting from 1 (not 0). How could they represent those numbers visually? If they need a hint, draw the first two figures in the following sequence:



Introduce the idea of a pattern rule. How can we express mathematically the growth that is happening in the triangle and the 'Ls'? Mention that these simple rules of plus 1 and plus 2 will take on significant development in formal algebra.

Distribute copies of the Pascal's triangle worksheet. Have Student Teachers work independently to identify both repeating and growth patterns. If they need a prompt, suggest that they look on the diagonals.

To end the session, have Student Teachers share and discuss the patterns that they found. Remind Student Teachers that just as they were challenged by this activity of discerning patterns in a complex numerical and visual format such as Pascal's triangle, young children can be equally confused when faced with what appears simple to adults: an AB AB pattern of coloured cubes.

Assignment

To prepare for the next class session, Student Teachers should read the following article 'What Do Students Struggle with When First Introduced to Algebra Symbols?', available as a PDF:

- http://tinyurl.com/Struggle-Algebra

Unit 2/week 1, session 2: Algebra as generalized arithmetic

What do Student Teachers need to know?

Algebra is a symbolic way to express what students already know from their work with arithmetic, numbers, and operations.

The symbol '*x*' is not only an unknown in equations but also a variable in expressions.

Equivalent formulae can be expressed in different formats.

Adults' past experiences learning algebra affect their definition of algebra and how it is taught. Addressing these experiences may require unlearning images of algebra and replacing them with images of broader algebraic thinking rather than just symbolic notation and algebraic formulae.

How do children think about these concepts?

From the article 'What Do Students Struggle with When First Introduced to Algebra Symbols?': 'It was found that only a small percentage of students were able to consider algebraic letters as generalized numbers or as variables, with the majority interpreting letters as specific unknowns'.

Children's formal introduction to algebra usually consists of finding x as the unknown, a particular answer for a given equation. However, x is not only an unknown. It can also represent a variable where, for example, there is an infinite number of answers for $y = x + 1$. This distinction between the unknown and the variable is crucial for Student Teachers to understand.

Children think the terms x and y are somehow both mysterious and unchangeable. A school principal once asked me, 'Why x and y ?' To which I responded that x and y could be any letters (sometimes n) and that they were purely a mathematical convention. I also noted that x usually represents the independent variable (the input), whereas y usually represents the dependent variable (the output).

Children usually are not clear about the relationship among equivalent relationships, such as these descriptions for the area of a square. Children tend to see them as totally different.

- Measurement from a numerical formula (the area of a square with a side length of 3 can be expressed as 3 x 3 = 9)
- A symbolic formula (A = s x s)
- The algebraic formula for the area of a square expressed in x and y notation, $y = x^2$

Note the difference between the visual orientation for notation in algebra and arithmetic. In most arithmetic equations, the answer comes 'to the right'. Instead, most algebraic formulae have the dependent variable y (the answer) on the left.

Later, when y is replaced by function notation $f(x)$, students become confused, thinking that this means 'f' is multiplied by x, as they recently have been introduced to the algebraic format for multiplication, such as $2x$ or ab.

What is essential to do with Student Teachers?

- Have a small group, then whole-class, discussion about the article 'What Do Students Struggle with When First Introduced to Algebra Symbols?'
- Have Student Teachers work with familiar algebraic equations by using n as a variable to develop patterns that illustrate how algebra can be thought of as generalized arithmetic.

Activities with Student Teachers

Begin the session by dividing Student Teachers into small groups to discuss 'What Do Students Struggle with When First Introduced to Algebra Symbols?', the article read for homework. Assign each group to read the first question and one of the other discussion prompts on the accompanying worksheet. Ask them to discuss both the article as a whole and the questions their group has been given. Allow about 8 to 10 minutes for the small group discussion.

- How does this article relate to the way you learned algebra as a student? How can x be both the unknown and a variable?
- What was your first instinct when solving the word problem about sharing money? Did you use arithmetic or algebra? What difference do you see between these two methods? How might you help children begin to shift from arithmetic to algebraic ways of thinking?
- When you saw the two-column chart and looked for patterns, what did you see? What type of thinking led children to come up with other patterns? Were those patterns valid? Why or why not?
- When you saw the growth pattern in Figure 2, how did you extend the pattern to find the number of sticks in Figure 25?
- What do you think of the author's comment that 'moving from arithmetic to algebraic generalizations is a process that has been found to take time'. If this is so, when should the 'algebraicifacation' of arithmetic begin?
- How does the difference between the minus sign for the operation of subtraction and the negative sign for numbers less than 0 influence students' work in algebra?

Have a large group discussion, starting with Student Teachers' overall thoughts of how they were taught algebra. Did they 'struggle when first introduced to algebraic symbols'? Was algebra connected to arithmetic when they began studying it? At what point did they notice a connection? Did they see algebra as generalized arithmetic or as equations into which they substituted numbers in order to come up with an answer?

Ask what they thought of the idea of x as a variable, not just as an unknown. Was this a new idea for them?

Then have each group share their thoughts for the other question they were assigned. Ask for other groups' input on the ideas expressed.

To end the discussion, note that all these questions are fundamental to the teaching of algebra and ask if algebraic thinking begins in the early grades. Which model for x (unknown or variable) might younger children find easier to understand? How, if teaching upper-grade students, would they integrate these two models?

For the remainder of the class, have Student Teachers investigate the link between algebra and arithmetic. In secondary school most students were given generalized equations such as $(n + 1) \times (n - 1) = n^2 - 1$. Have them create a five-column chart with the column headings:

- n (the numbers 1 through 10)
- The expression $(n + 1) \times (n - 1)$
- The arithmetic computation using n in the expression $(n + 1) \times (n - 1)$
- The expression $n^2 - 1$
- The arithmetic computation using n in the expression $n^2 - 1$

What do they notice?

Have Student Teachers create a second chart for $(n + 1) \times (n + 1) = n^2 + 2n + 1$. Why does $(8 + 1) \times (8 + 1) = 64 + 16 + 1$, or 9^2, or 81? Does this pattern hold true for all the numbers in their chart? Would this growth pattern continue for all *n*s?

End the session by having Student Teachers discuss how this activity relates to the idea of x as a variable and how algebra can be thought of as generalized arithmetic.

Assignment

Have Student Teachers work the problem on the difference of squares from this website:

➢ http://tinyurl.com/Diff-of-Squares

Have them read the article 'Algebra in the Elementary Grades? Absolutely!', available at:

➢ http://tinyurl.com/M-Burns-Articles

Unit 2/week 1, session 3: The algebraic thinking of young children

What do Student Teachers need to know?

Children can be introduced to growth patterns at an early age and can learn to extend growth patterns.

A problem can give rise to equivalent expressions describing correct solutions.

Knowing how to use the distributive property of multiplication over addition is often a key strategy to understanding why expressions are equivalent.

Teachers can monitor young children's algebraic thinking and use it to highlight important algebraic generalizations.

How do children think about these concepts?

As mentioned earlier, young children notice repeating patterns earlier than growth patterns. Because growth patterns are less obvious to them, children benefit from working with real objects before being asked to represent the pattern by drawing.

It is also important that the teacher listen to individual children's way of describing the pattern, as it is likely that they will have different methods for thinking about the problem.

Having children chart their numerical findings on a T-chart is another way to represent the growth pattern.

The idea of multiple representations for the same algebraic function will continue into later grades when children will be able to use graphs as a another representation.

The distributive property of multiplication over addition is not something that is usually formally introduced to young children. It is likely, however, that children can make informal use of the distributive property in early algebra activities.

Helping children articulate what they are doing in these informal situations paves the way for their having a firm sense of the distributive property when it is formally introduced in later years.

What is essential to do with Student Teachers?

- Begin the session by referring to the article Student Teachers read for homework, 'Algebra in the Elementary Grades? Absolutely!' Have them refer to the article's format, the author's expositional text, and the author's commentaries on students' thinking.
- Have Student Teachers engage in the article's activity, extending a growth pattern and coming up with a pattern rule.
- Note the distributive property of multiplication over addition and the idea of equivalency.
- Discuss what they learned about children's mathematical thinking from reading the article.

Activities with Student Teachers

Begin the session by discussing the article read for homework, 'Algebra in the Elementary Grades? Absolutely!'

Mention that this is not simply an article that presents student work for analysis but that it provides insight into seven-year-old children's thinking as they are working on an algebraic activity involving patterns.

As the Student Teachers refer to the article, have them note the role of the teacher. What does the teacher notice? What does she say? How might she use what she sees and hears during student work time to have a discussion at the end of class to bring students' ideas into sharper focus?

Although you might not have enough coloured blocks for Student Teachers to simulate the problem, you can draw the first two examples on the board and then have them create patterns of trees on blank paper.

Note that this is really a growing pattern because it's a simulation of how a small tree grows from year to year. Notice how the width of each year's growth layer decreases as the years go by.

After Student Teachers have drawn about 10 years' growth for their trees, ask them to stop and consider a mathematical expression for the tree's growth; e.g. for x = years (x trunk sections + x leaf sections + the top triangle). Ask them how many shapes they drew for any particular year.

The summary is an opportunity to discuss three important mathematical concepts:

- The idea of a pattern rule
- The distributive property of multiplication over addition [x (trunk shapes + leaf shapes) + 1 top shape]
- The idea of equivalent expressions, that [x (trunk shapes + leaf shapes) + 1 top shape] is equivalent to (x trunk shapes + x leaf shapes + 1 top shape)

Early attention to equivalence and equivalent expressions is important because it will be featured in later sessions of this unit.

End the session by having Student Teachers refer back to the article and read through the photo scenarios of the teacher's observation of individual students. What was the teacher thinking about student thinking?

Ask Student Teachers to consider if they were the teacher in this classroom and heard all these student ideas, how would they prepare an end-of-class summary that moved from rudimentary to more sophisticated thinking about growth patterns?

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 2/week 2: Variables, coordinate graphs, multiple representations, equivalence

Session 1: Developing an understanding of x as an unknown and as a variable, coordinate graphing, discrete graphs
Session 2: The importance of multiple representations in algebra, continuous graphs
Session 3: Introduction to symbolic representation and mathematical equivalence

Faculty preparation for the upcoming week (1–2 hours)

- Read the following articles and look at these websites that address variables, multiple representations, coordinate graphs, and equivalence:
  - The *College Preparatory Mathematics* teaching guide for its 'Connections with Multiple Representations' chapter. You'll note on page three that secondary students are given the same tree growth problem that seven-year-olds worked on in the article last week. There are also excellent diagrams of growth patterns linked to how they can be represented in various formats:
    - ➢ http://www.cpm.org/pdfs/information/conference/AC_Con_Mult_Rep.pdf

    Also available at:
    - ➢ http://tinyurl.com/Algebra-Mult-Rep
  - The introduction to Mark Driscoll's *Fostering Algebraic Thinking*. Note the list of questions on page three that you can use as discussion prompts during class to raise issues about the teaching of algebra with Student Teachers:
    - ➢ http://www.heinemann.com/shared/onlineresources/e00154/intro.pdf

    Also available at:
    - ➢ http://tinyurl.com/Fostering-Algebra
  - Annenberg Learner's 'Insights into Algebra' online teacher education unit, 'Variables and Patterns of Change' (video):
    - ➢ http://tinyurl.com/Algebra-Insights
- Review the free online graphing calculator. This free online graphing calculator requires no subscription, no downloading, no software, and has no advertising. In line with the idea of multiple representations, this calculator also creates a table of values next to the graph:
  - ➢ http://tinyurl.com/Free-Graph-Calc
- Download and print out for your use:
  - 'The Coin Graph' activity (available as a resource in the Course Guide)
  - A colour transparency of solutions for 'Tiling the Pool':
    - ➢ http://tinyurl.com/Pool-Solutions
- Download and print out copies for Student Teacher use:
  - 'Kitchry (Moong Dal Rice) Recipe' (available as a handout in the Course Guide)
  - 'Graph Stories' (available as a resource in the Course Guide)
  - 'Tiling the Pool' (available as a resource in the Course Guide)

- Bring to class:
  - Coins (enough so that each Student Teacher can receive two)
  - Chart paper (ideally chart-sized graph paper) and markers
  - Graph paper
  - Rulers
  - Crayons or coloured pencils
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 starts with an introduction to x. Although most Student Teachers have been taught to think of x as the unknown to solve for in an equation, this course begins with another meaning for x: a variable. Student Teachers will engage in a short discussion about what they think this new interpretation of x means, leading to an emergent understanding of what variables are, when they are used, and how they express change.

Talking about variables in the abstract, however, is not particularly helpful to Student Teachers' understanding of what variables are, how they are used, and how they express change. In order to help make sense of variables, this session will also introduce coordinate graphs to help Student Teachers visualize how change in one variable affects a change in the other. The idea of direct variation will be introduced along with a definition of a discontinuous graph.

Student Teachers may also need to refresh their memory of how to set up a Quadrant I graph, label the graph, label the axes, and plot points. This will be one of the times that the Instructor will use the pedagogical strategy of whole-class demonstration, having Student Teachers co-create a table of values (T-chart) and coordinate graph.

Session 2 continues with the idea of how multiple representations of the same algebraic situation need to be recorded. After having explored a discrete graph in the prior session, the Student Teachers will now work with continuous graphs where a function is continuous.

The session will begin with Student Teachers looking at two graphs on the handout 'Graph Stories' and then being asked what they think these graphs mean. How are they different from discrete graphs? Both of the graphs on the handout involve the idea that time is continuous but that change over time is continuous only for a specific interval. The session will then move on to look at a graph of a linear function.

Session 3 will address the concept of equivalence by using the classic problem 'Tiling the Pool'. The focus here will be to integrate a narrative, drawing, table of values on a T-chart, and coordinate graph with the end goal of Student Teachers discovering a symbolic representation for a function or 'pattern rule'. Student Teachers will work in pairs to come up with a generalized function to describe the visual representation, again thinking of how to extend a growth pattern but this time by a function rule, not by a counting method. Note that there are multiple equivalent expressions that are solutions to this problem. (Some pre-algebra students have discovered 10 equivalent expressions!)

Unit 2/week 2, session 1: Variables and coordinate graphs

What do Student Teachers need to know?

Variables are symbols (often letters) used to represent patterns of change. The input variable is called the *independent variable*; the output variable is called the *dependent variable*.

Variables are used in pattern rules (later called *functions*) to indicate a relationship and a rate of change between variables.

Patterns of change can be represented on a coordinate graph that is created from data collected on a T-chart.

The coordinate plane is divided horizontally by the x-axis (for the independent variable) and vertically by the y-axis (for the dependent variable). This is why the columns of a T-chart are often labelled with the variables x and y.

How do children think about these concepts?

Usually a child is first introduced to algebra by being asked to find the value of x as the unknown, a unique answer for a given equation. However, x is not only the unknown, it can also represent a variable, as where there are infinite answers for the expression $x + 1$. The distinction between these two meanings for x is important for teachers to understand so that they can communicate this difference to children in their classrooms.

When setting up a graph, Student Teachers need to create a scale on each axis with a consistent interval between each cell on the grid. Children often 'scale the data' by using the numbers, which may not have a consistent interval between them, from their T-chart.

Children have a tendency to connect the points they have plotted on their graph. When changes are continuous (as for the circumference of a circle), the points should be connected to show that continuous change is happening between the data points. In other cases, often involving counting objects, there is no change between the data points and the student should not connect the points. This kind of graph is termed a *discrete graph*.

What is essential to do with Student Teachers?

- Introduce variables: what they are and when and how they are used.
- Remind Student Teachers how to set up a Quadrant I graph.
- Have Student Teachers collect data, record it on a T-chart, and then transfer that information to a coordinate graph.
    - Help Student Teachers understand how to:
        - Set the range and appropriate scale for each axis
        - Label each axis and title the graph
        - Plot the data points (x,y) from the chart they created.
- Introduce the idea of a pattern rule that can be used to describe the change between two variables.
- Introduce the concept of direct variation, in which the variables change at the same rate.

Activities with Student Teachers

Begin by having Student Teachers give examples of things that change over time. Ask how those changes might be represented. Student Teachers may, for example, refer to the tree problem from the previous session, where the number of blocks used changed according to the age of the tree.

Other changes—such as distance as a function of time related to speed ($d = rt$) or the change in a circle's circumference as a function of its diameter [$C = \text{pi}(d)$]—may be mentioned. Note that what students often call formulae are actually pattern rules put into equations.

Use 'The Coin Graph' activity. Do this activity as a whole-class demonstration while Student Teachers create the same T-chart and graph.

During the coin-collecting activity, create a T-chart of the data, adding x and y to the names of the two columns. Note that x is the independent variable and y is the dependent variable.

When this activity is complete, refer to the empty grid paper on which you and they will create a graph together. Demonstrate how to set up a Quadrant I coordinate graph. Note that you labelled the axes to reflect issues for this particular activity but that the conventional way of discussing the two axes is to call the horizontal axis x and vertical axis y.

Next, ask how you should add a scale to each axis. What would be a reasonable interval to gather all the data from the T-chart on to the graph? Proceed to label the axes with numbers.

Have Student Teachers come up to the graph, refer to their T-chart, and plot the points. When finished, ask about a latecomer scenario. How would the graph look if someone added their coins after class was in session, extending the table and graph? What if this activity were done in an auditorium with many more people? How would that affect how they might scale the graph?

What about a pattern rule? What in the table and on the graph suggest one? Is the pattern rule the same for both the table and the graph?

Note that this activity involved what is called *direct variation*, in which the numbers changed at a constant rate. Mention that not all change between two variables is constant.

Ask Student Teachers if they think they should connect the points on the graph. They should not. This will give rise to the difference between discrete and continuous graphs.

End the session by noting that in the next session they will continue to explore various ways to represent algebraic data and that they will look at data that are not discrete but continuous.

Assignment

To be determined by the Instructor.

Unit 2/week 2, session 2: Continuous graphs, multiple representations

What do Student Teachers need to know?

The appearance of a graph can imply a story of change.

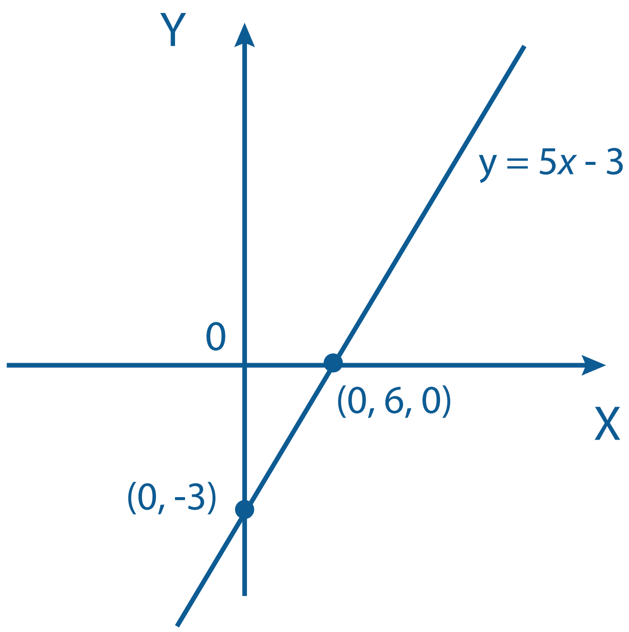
Change on a graph is shown in certain intervals.

Some continuous graphs are composed of only one straight line, which shows a linear function.

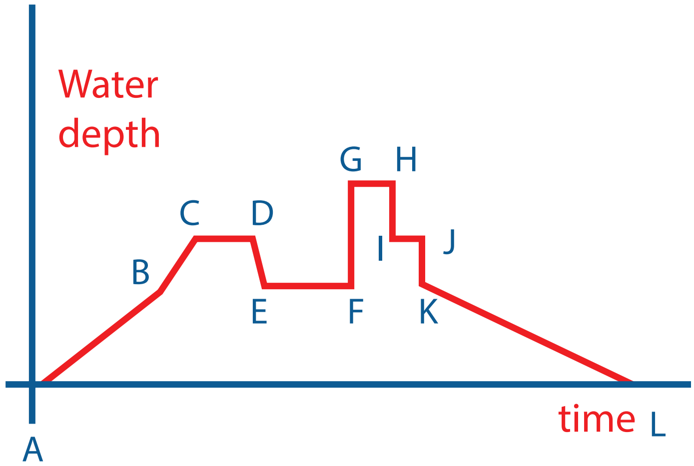
Tables can also show a linear function by a constant 'first difference' between consecutive *y* values.

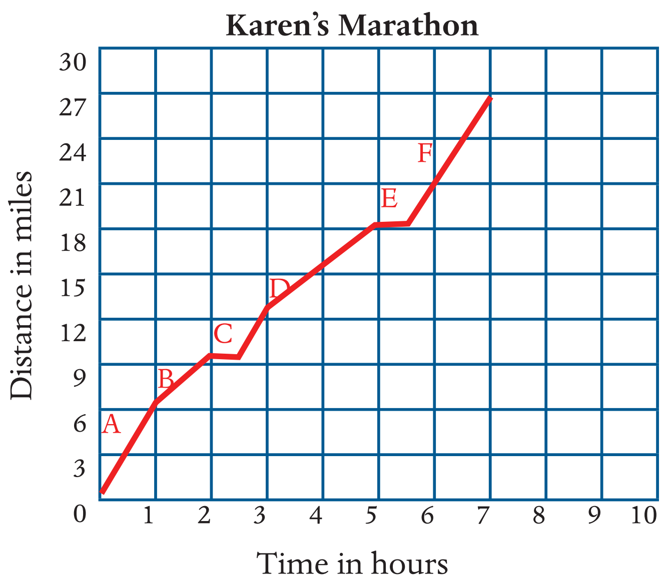
How do children think about these concepts?

Many children think that graphs are simply a mathematical arrangement of points and lines on a grid, done for their maths teacher and created in a procedural manner. To help them expand their thinking about graphs, it is important that they analyse graphs that do not look like this abstract textbook illustration (which is not even presented on a coordinate plane):



Using graphs such as the water level and marathon graphs (in the handout 'Graph Stories') allows them to think about how a graph can be interpreted to tell a story of real-world events.





Similarly, because algebraic graphing is often introduced in a unit on linear functions, children may assume that algebraic graphing is just a matter of drawing a straight line between two points and that all algebraic graphs are line graphs.

By using multiple forms of representation, children who formerly had this vision of algebraic graphs can begin to see important connections between diagrams, tables, graphs, equations, and written and oral narratives.

Although all of these representations can model the mathematics in real-world situations, the most important reason for using multiple representations is to illustrate a mathematical connection. When children realize this, they begin to understand the power of algebra more deeply.

When entering data into a table of values (T-chart), children often create column headings that use the first initial of the variable. Thus, when exploring the relationship between the side length of a square and its area, they may label the first column 's' and the second column 'A'. At some point children need to generalize the idea of the variable and understand why those columns are conventionally labelled x and y.

This becomes especially important when children have access to graphing calculators (either hand-held or web based), because calculators only accept x and y as variables (unless a hand-held one has been programmed with a specific formula).

What is essential to do with Student Teachers?

- Introduce the idea that a graph can be interpreted to tell a story of change over time.
- Have Student Teachers use the real-world context of increasing and decreasing ingredients in a recipe to illustrate the difference between a discrete and continuous graph.
- Have Student Teachers develop a symbolic expression that could show the relationship between x and y.
- Have Student Teachers discuss the connections among all the various representations used in solving the problem.

Activities with Student Teachers

Begin the session by distributing the handout of the water level and marathon graphs, 'Graph Stories'.

Ask Student Teachers how these differ from the discrete graphs they explored in the previous session. Noting that they probably learned about line graphs as secondary school students, ask if these two graphs could be considered line graphs. Have them consider that each of these is a graph composed of several line segments, each constituting something that was continuous but which happened within a particular interval or time frame.

Ask them to pose a timeline about what different lines in the graph of water in a tub might mean. Why does it begin and end at 0? What might account for the fall and rise in the depth of the water?

When analysing the marathon graph, draw attention to question 2, which asks about what happened during Interval C. What does a horizontal line on a graph mean?

Distribute the handout 'Kitchry (Moong Dal Rice) Recipe', which asks Student Teachers to consider how the recipe will change if they need to decrease or increase each ingredient to serve different numbers of people. Have Student Teachers chart and graph two different types of ingredients: the moong dal (a measurement expressed in grams) and the cardamom (expressed as a countable item). Can they predict what each graph will look like?

When they have completed the assignment, ask questions to help Student Teachers analyse and compare the two graphs. Then compare them to the coin, water level, and marathon graphs.

Can the Student Teachers have predicted what the graph would look like by using the table? What patterns do they notice? Introduce the idea of *first difference*, the difference between successive values for *y*. Note the moong dal graph is called a *linear function*, whereas the graph describing the data is a single straight line.

Finally, ask Student Teachers to write an expression in symbolic terms that could serve as a rule for this linear relationship.

End the session by referring to the use of multiple representations (narrative, table, graph, and expression) that were used to solve the problem. Ask what connections they saw among these different approaches to solving the problem. Ask what they thought were the advantages and disadvantages of each representation.

Ask which one of these representations seemed most useful. If Student Teachers offer a strategy that relies more on arithmetic than algebra, try to steer the conversation toward an algebraic approach.

Assignment

Have Student Teachers experiment with this free online calculator to see how the work they did by hand today is interpreted by technology:

- http://tinyurl.com/Graph-Calc-Free

Why does their cardamom graph (a discrete graph) show up as a line when using the calculator?

Unit 2/week 2, session 3: Equivalence, multiple representations

What do Student Teachers need to know?

The idea of the equals sign, discussed in Week 1, Session 3 of the 'Numbers and Operations' unit, takes on new and extended meaning in this algebra unit, where symbols (not just numbers) are added to Student Teachers' thinking about equivalence.

Not only can numerical equations such as $3 + 5 = 7 + 1$ be proved to show equivalence, so can algebraic equations such as $4s + 4 = 4\ (s + 1)$.

This idea of determining proof of equivalence, justified by symbol manipulation, is one of the cornerstones of algebra, moving young children's informal algebraic thinking to more formal ways to think about equivalence.

Multiple representations of a problem show how a table of values and its corresponding graph can give rise to different symbolic expressions that are equivalent to each other.

The distributive property of multiplication over addition becomes an important solution strategy when comparing expressions to investigate their equivalence.

Symbolic representation allows children to move from the calculation of specific data to a generalized formula that can be expressed in variables.

How do children think about these concepts?

When children are given the problem of tiling a pool with a side length of 5, usually their first instinct is to assume that the solution must be either:

1) the perimeter surrounding the border tiles (28) or
2) the perimeter of the pool (20) (not considering that there are 4 corner tiles).

Because of what they have learned about perimeter in earlier grades, they often think that one of these two lengths can be translated into the number of tiles surrounding the pool.

Teachers need to be aware of and anticipate such common responses so that they can sensitively redirect a child's thinking to the question posed.

As children progress through the elementary grades, they will develop a formal sense of the distributive property of multiplication over addition. They will need to be reminded of the informal way they have used the distributive property in earlier grades and then be directed to the procedural way in which it is applied when using symbols.

In addition to the use of multiple representations, the 'tiling the pool' problem involves the informal use of symbolic notation and symbol manipulation. Research shows that students need informal opportunities to solve problems involving symbols and symbol manipulation before being introduced to these procedures in a formal manner.

What is essential to do with Student Teachers?

- Have Student Teachers solve the 'tiling the pool' problem, which involves a growth pattern, by using multiple representations as solution strategies. After working with one-digit numbers for the side length of the pool, can they extend their pattern rule to a pool with a side of 50 units? For a pool with a side length of unspecified units?
- Have Student Teachers represent a variety of solutions in symbolic format, then use symbol manipulation to demonstrate that all their correct solutions were equivalent.
- Introduce the idea that although all their solutions were equivalent, they might not look the same if the problem was solved without a corresponding coloured diagram. Use the coloured transparency with solutions to emphasize that while solutions may be mathematically equivalent, they are not necessarily the same in a real-world situation.
- Have Student Teachers connect the various representations they used in solving this problem.

Activities with Student Teachers

Introduce the 'tiling the pool' problem by distributing the handout of the same name. Refer to the directions, answer any general questions, and have Student Teachers work in pairs to solve the problem.

Allow plenty of time for this, as Student Teachers will be creating diagrams, table, graph, and several symbolic expressions that all represent the same scenario.

Listen to their conversations during this assignment as they pose tentative ideas and resolve them.

After Student Teachers have finished finding symbolic expressions, bring the group together and ask about the symbolic expressions that their pictures, table, and graph helped them discover. Have Student Teachers report their expressions and note them on the board. Some Student Teachers' contributions may be:

$$4s + 4$$
$$4\,(s + 1)$$
$$2s + 2(s + 2)$$
$$2s + 2s + 4$$
$$4(s + 2) - 4$$
$$s + s + s + s + 4$$

Ask how these (correct) expressions, which all look different, can be proved equal. If someone mentions the distributive property, follow through on this. If not, this is the time to formally introduce the concept. If a Student Teacher suggests an expression that is not on the above list, have them try to show equivalence.

Ask if they saw 'first differences' in their table. How did that contribute to how they envisioned the graph?

If no one has suggested it, ask about the following expression:

$$(s + 2)^2 - s^2$$

Is this expression equivalent to those listed above? (It is.) Moreover, how can a quadratic expression involving two square numbers result in a linear solution?

Ask Student Teachers to think decoratively. What would some of the above patterns look like if a designer used more than one colour tile for the border? Ask how the following expressions (tile designs) would look in colour. Have Student Teachers draw three 4 x 4 pools plus their surrounding tiles on graph paper. Have them colour in a border that could show these three different border designs:

$$n = 4s + 4$$
$$n = 4(s + 1)$$
$$n = 2s + 2(s + 2)$$

Use the colour transparency of solutions to help highlight how various expressions look different when coloured.

End the session by asking how this week's focus on multiple representations has influenced their own algebraic thinking. How might this have changed their assumption as to how they would introduce algebraic concepts, graphing, and multiple representations to their future students?

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 2/week 3: Linear functions, slope, order of operations

Session 1: Introduction to linear functions
Session 2: Introduction to the concept of slope
Session 3: Introduction to the conventional order of operations

Faculty preparation for the upcoming week (1–2 hours)

- Look through the following website on linear functions and order of operations: 'Insights into Algebra' online teacher education unit (from Annenberg Learner), 'Linear Functions and Inequalities':
  - ➢ http://tinyurl.com/Insights-Linear
- Download and print out for Student Teacher use the following handouts:
  - 'Taxi Fares' (available as a resource in the Course Guide)
  - 'Stairs according to Code':
    - ➢ http://tinyurl.com/Algebra-Stairs
  - 'Counting for Slope':
    - ➢ http://tinyurl.com/Just-Slope
  - 'Rise-Run Triangles'
    - ➢ http://tinyurl.com/Rise-Run-Triangles
- Bring to class:
  - Graph paper
  - Rulers
  - Crayons or coloured pencils
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 builds on last week's introduction to linear functions, briefly reviewing the characteristics of the table, graph, and symbolic representation of the recipe problem.

Student Teachers will be guided to notice that when using conventional x and y notation, their equation resulted in the generalized equation of $y = \mathrm{m}x$, where m was a multiplier of ingredients in the original recipe.

It is also important that they understand that the multiplier (m) can be not only a whole number but a fraction, decimal, or mixed number.

After this review, Student Teachers will engage in a problem in which the y-intercept (which has not been formally discussed) is not 0. The informal definition of the y-intercept will be that of a starting point, such as in a problem involving finances where there is a basic initial cost.

For example, a taxi in Dubai imposes an initial cost for a ride. A cost is then added for each fraction of a mile travelled:

- 6 Dirhams upon entry into the taxi
- 1.5 Dirhams for each one-fifth of a mile travelled

This can be modelled by a linear equation where the starting cost (the y-intercept) is 6, to which is added $1.5x$, with x being the number of kilometres travelled.

Although there are certain problems, such as the tiling the pool problem, that lend themselves well to diagrammatic representations, the taxi problem is more likely to be solved by talking through it and then representing it in a table and graph before coming up with an equation.

This taxi problem, like the recipe problem, is an introduction to linear functions set in a real-life context to which adults such as Student Teachers can relate. Finding linear function problems for students in lower secondary grades can be challenging. However, it is important that they also understand linear functions more deeply by solving problems that are meaningful to them.

Session 2 will emphasize how linear equations in the form of $y = \mathrm{m}x$ are related to slope. The idea of 'first difference' mentioned earlier will become important, as it relates to the conventional formula for slope, which is expressed as the 'difference in y related to the difference in x'.

Mentioning first difference in last week's session sets a foundation for Student Teachers' understanding of how subtraction in the following symbolic representation is related to the first differences they noted on their chart.

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Session 3 begins by looking at the conventional order of operations in arithmetic and the use of clarifying devices such as parentheses when evaluating more complex algebraic expressions. As usual, this will be approached not by presenting a series of rules but by giving Student Teachers an arithmetic problem requiring several operations that may perplex them, resulting in more than one answer. When discussing their answers, Student Teachers will be introduced to the conventional order of operations and then move to discover how algebra uses additional notation to make the order of operations much more clear.

Unit 2/week 3, session 1: Linear functions

What do Student Teachers need to know?

There is a connection between the various representations for linear functions.

All linear functions have certain characteristics in common: a straight-line graph, a constant difference on a table, and an equation written in the form of $y = mx + b$.

The 'b' in $y = mx + b$ is the y-intercept on a graph.

You can recognize a linear function simply by looking at a graph, table, or an expression.

Differences in the appearance of a graph showing the same data points are due to a difference in scale.

How do children think about these concepts?

When introducing linear relationships, textbooks often present the formula $y = mx + b$, give a table of values, and then ask children to create a graph.

In this course, we take the opposite approach:

- Learn about a table of values by creating what is called a T-chart
- Create a graph by using the table of values

Only when these two skills are in place do we introduce symbolic notation. This sequence of events helps students make sense of the equation for a linear function, $y = mx + b$.

When presented with a linear function problem where the y-intercept is not 0, middle grade students are often surprised or confused. This is because the emphasis in their prior work may have been with situations where graphs began at (0,0).

Similarly, if children have worked only with Quadrant I graphs, they will need the teacher's help so that they can envision the entire coordinate plane. This is where their integer work in the 'Numbers and Operations' unit with horizontal and vertical number lines can help Student Teachers connect with a four-quadrant graph.

What is essential to do with Student Teachers?

- Introduce a linear function problem in which the y-intercept is not 0.
- Have Student Teachers create a table, graph, and symbolic expression to represent fares over various distances.
- Have a whole-class discussion that brings out similarities and differences between the taxi problem and the recipe problem from last week.
- Introduce the conventional format for linear equations: $y = mx + b$.

Activities with Student Teachers

Begin the session by distributing copies of 'Taxi Fares' and ask Student Teachers to work in pairs to create a table, chart, and symbolic expression to represent fares over various distances.

Note how Student Teachers set up their graphs. Do they scale the y-axis in whole number amounts? What interval did they use for distances, which are expressed in 'one-fifth of a mile' units? To how many miles did they extend their table of values?

It is important that pairs of Student Teachers make these decisions independently and that you do not give them specific requirements. This is to illustrate later that the same data can look different when plotted on graphs that are scaled differently.

When Student Teachers have finished the assignment, begin a discussion about the look of each graph. Are they all the same? If not, how are they different? Why is this so?

Have Student Teachers recall the recipe problem from last week, asking how the taxi problem is similar to it.

Student Teachers should note that the lentils graph showed a straight line, the table showed a constant first difference, and the expression was in the form of mx.

Ask how the two problems differ.

Continue by discussing that because there is an initial fee, the graph begins at 6 on the y-axis. This is different from most graphs they have encountered, which begin at (0,0).

Tell Student Teachers that there is a special name for the place where a line crosses the y-axis: the y-intercept.

Ask how this is represented in the expression they wrote. Remind Student Teachers that the expression they wrote for the recipe problem last week was in the form of $y = mx$.

What is the form of the taxi problem's expression? Ask if there is a connection between the y-intercept on their graph and the 6 in their expression.

Note that in the recipe problem the graph started at 0, so the expression could be thought of as $mx + 0$. However, in the taxi problem, the expression is $mx + 6$. Tell Student Teachers that the conventional notation for all linear equations is $y = mx + b$, where b is a constant and represents the y-intercept.

Assignment

To be determined by the Instructor.

Unit 2/week 3, session 2: Linear relationships and slope

What do Student Teachers need to know?

Slope is a characteristic of linear functions. It is the constant rate between two variables.

In a linear equation in the form of $y = mx + b$, the slope is the coefficient m.

Slope is informally described as 'rise over run'. It can be more formally thought of as 'the vertical change divided by ("over") the horizontal change', or expressed by the following formula:

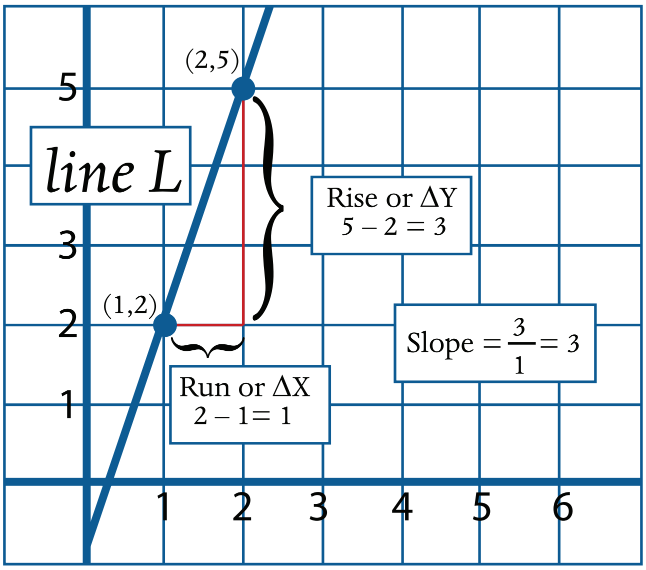
$$\frac{y_2 - y_1}{x_2 - x_1}$$

Slope implies steepness, but the same slope on graphs scaled differently may look different.

The slope of a line can be positive (going up), negative (going down), or 0 (a horizontal line where there is no change).

How do children think about these concepts?

In many textbooks, slope is introduced via a 'rise-over-run' visual on a graph. Although the following diagram may seem self-explanatory to adults, it is too abstract to be useful in introducing slope to middle grade students.



Children eventually will need to interpret this visual explanation of slope, but it has too many terms and symbols to be an effective introduction to the concept.

Although children usually interpret slope as a measure of steepness, the scale of a graph can be misleading. Graphs with different scales will present different visual images of steepness. This is also true of most graphing calculators, where the cell on the graph is rectangular, not square. The rectangular cell means that a line with a slope of 1 does not rise at a 45-degree angle to match the graph that a child created on graph paper on which the cells are square.

Children need to connect the concept of rise to the vertical distance on the *y*-axis and *run* to the horizontal distance on the *x*-axis to understand the conventional formula for finding slope (shown below). However, teachers often assume that children can translate this 'rise over run' into the formula 'delta *y* over delta *x*'.

Children are usually confused as to what delta means and how *delta* relates to the changes in *y* and *x* in the conventional formula for calculating slope:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

What is essential to do with Student Teachers?

- Introduce the concept of slope by having Student Teachers work with the 'Stairs according to Code' problem.
- Have Student Teachers discuss the features of their graphs that involve slope.
- Clarify the difference between a graph's steepness and its slope.
- Introduce the term *coefficient* and point out its role in indicating slope (m) in $y = mx + b$.
- Introduce three ways to talk about slope:
  1) Rise over run
  2) Change in vertical versus horizontal change
  3) This formula:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

- Introduce several equations in $y = mx + b$ format that have a positive slope, negative slope, and negative *y*-intercept.

Activities with Student Teachers

Begin by distributing copies of the 'Stairs according to Code' problem. Have Student Teachers follow the directions, first by creating a graph, then making a table of values, and finally developing an equation in $y = mx + b$ format. (Again, do not give parameters for the graph, so that Student Teachers will create graphs with different scales that can be used for comparison.)

Once Student Teachers have completed the assignment, have them discuss their graphs in informal terms, noting that the ratio (slope) between any two points is 7 'up' versus 10 'over', or 7/10. (It is important that Student Teachers realize that the same 7 to 10 ratio exists between any two points on the line, not just two adjacent points.) Have Student Teachers use crayons or coloured pencils to colour in a slope triangle made by the line.

Because students usually equate slope with steepness, point out the different appearances of the graphs they created. Why does the same data have different steepness but the same slope?

When you move to the symbolic expressions that Student Teachers developed, introduce the term *coefficient*, relating it to the 'm' in the equation $y = mx + b$.

This completes Student Teachers' introduction to $y = mx + b$: b is the y-intercept (a constant), m is the slope of the line, x is the input (independent) variable, and y is the output (dependent) variable.

Help Student Teachers see the connection between 'rise over run' and 'vertical change divided by horizontal change' before introducing the following formula. (Do not assume that they will have a clear understanding of the connection among all three ways to express slope by the end of the lesson.)

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Present the following set of equations in $y = mx + b$ format. Ask Student Teachers to predict what the graph of each will look like. Note that one of the equations has a negative slope whereas another has a negative y-intercept. Do not attempt to go into depth about these two new ways to think about linear relationships on the coordinate plane, but have Student Teachers use the free online calculator to explore these four equations before the next class.

$$y = 5x$$
$$y = 5x + 2$$
$$y = 5x - 2$$
$$y = -5x + 2$$

Assignment

Have Student Teachers use the 'Rise-Run Triangles' handout to explore slope.

Have Student Teachers use the 'Just Slope' handout to explore linear relationships that result in negative slopes.

Have Student Teachers use the free online calculator to explore these four equations before the next class.

$$y = 5x$$
$$y = 5x + 2$$
$$y = 5x - 2$$
$$y = -5x + 2$$

Unit 2/week 3, session 3: Order of operations

What do Student Teachers need to know?

There are conventions when multiple operations are required to evaluate an expression.

These conventions are made easier in algebra by using parentheses to clarify which operations need to be done before others.

The context of a given problem can provide clues for using the order of operations to model the problem.

How do children think about these concepts?

The order of operations can be strange and confusing to children. They assume that they should calculate all numbers from left to right.

Because most basic calculators calculate numbers in sequential order from left the right, the resulting answer is often incorrect for a given problem that includes multiplication and division. This only strengthens children's belief that this 'left-to-right' method must be correct because 'the calculator says so'.

When told the correct order of operations, children either do not remember it or they rely on a mnemonic such as PEMDAS (or 'Please Excuse My Dear Aunt Sally') to recall that they should first operate within *p*arentheses, then address *e*xponents, *m*ultiply and *d*ivide, and only then *a*dd and *s*ubtract. Remembering a mnemonic, however, is purely procedural and does not help children understand why the conventional order of operations is crucial not just in basic calculations but also in real-world situations.

Once parentheses are introduced into an expression, calculation using the order of operations is much easier for children to understand and use.

What is essential to do with Student Teachers?

- Review the homework assignment that helped Student Teachers use slope triangles to visualize slope, especially negative slope and negative y-intercepts. Have Student Teachers develop an equation in $y = \mathrm{m}x + \mathrm{b}$ form for several of the graphs.
- Have Student Teachers work in pairs to solve an arithmetic equation designed to assess their understanding of the conventional order of operations without parentheses.
- Introduce the conventional order of operations. Then add how parentheses can clarify how to evaluate expressions that appear confusing.
- Have Student Teachers work on a problem that involves order of operations involving a real-life situation, then generalize this by using variables.

Activities with Student Teachers

Begin class by having Student Teachers share what they learned by working on the homework assignment. As the assignment focused only on the graph, extend the conversation by asking Student Teachers to consider how they might use the slope they discovered and the y-intercept to create an equation in the form of $y = \mathrm{m}x + \mathrm{b}$.

Do a few of these together, then have them see how quickly they can write equations for the rest of the graphs on the page.

Move to this session's focus topic, order of operations, by writing the following expression on the board: 5 x 8 + 6 ÷ 6 – 12 x 2. Have Student Teachers work in pairs to evaluate it. (The correct answer is 17.)

As they work on the problem, notice how they engage with their partner. What rationales do they have for the way they think the expression should be evaluated? Are Student Teachers negotiating with each other as to how to proceed? For Student Teachers who move from left to right, their answer will be 0.66666. (This is the answer that entering the numbers sequentially into a basic calculator would give—which is not the correct answer!)

Notice how long Student Teachers work to solve the problem. Student Teachers operating from the sequential left-to-right assumption will likely take more time as they will be working with fractions or decimals, as opposed to Student Teachers who are working with integers.

Ask Student Teachers to share their answers and how they determined them. This should give rise to several alternative ways of thinking about the problem. Ask why this occurred.

Honour the fact that all Student Teachers were clever and resourceful when thinking about how to address such a confusing calculation, but that some of their procedures did not follow the conventional way of addressing these types of equations.

At this point, tell Student Teachers that there is a conventional way to address these types of equations: order of operations.

Introduce the idea that order of operations insists on doing multiplication and division first. Only then can we add and subtract. Have Student Teachers use this information to re-evaluate the expression so that it results in 40 + 1 – 24.

Proceed to say that algebra uses a particular notation—parentheses—to make all of this easier. Parentheses group certain numbers and operations together so it is clear which operation to perform first. By using parentheses, the above expression becomes (5 x 8) + (6 ÷ 6) – (12 x 2). Make clear to Student Teachers that when looking at an expression, they need to perform the operation in the parentheses before doing anything else. Then they would do any remaining multiplication and division, and finally any addition and subtraction.

(At this point do not mention how the order of operations addresses exponents. This will be included in next week's focus on quadratic [square] equations.)

Present the following equation where C is cost: C = 10,000 + 15 x 2000. How would Student Teachers calculate the cost given what they now know about order of operations?

Extend the discussion about C = 10,000 + 15 x 2000 in several ways. First have Student Teachers insert parentheses to make the order of operations clear.

Next, ask, given what they now know about start-up costs from the taxi problem, which real-world situation this equation might refer to (e.g. a gathering where there was a Rs10,000 room rental fee plus dinner for 15 people at Rs2000 a piece). Just as a graph can tell a story, so too can an equation.

Finally, ask Student Teachers to disregard the specifics of C as cost and translate the above equation into the generalized $y = mx + b$ format. How could this same equation be used to calculate an initial deposit of Rs10,000 into a bank account with Rs15 deposited every week for 20 weeks?

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 2/week 4: Quadratics, solving for *x* as the unknown

Session 1: Introduction to quadratics I – Tables and graphs
Session 2: Introduction to quadratics II – Equations, connecting algebra and arithmetic
Session 3: Finding *x*, the unknown

Faculty preparation for the upcoming week (1–2 hours)

- Read the following article about students' understanding of *x*, the unknown:
  - What Is That '*x*' Anyway?' by D. M. Nguyen:
    - ➢ http://tinyurl.com/What-is-That-x
- Look through these websites that address quadratic equations and finding *x*, the unknown:
  - Teacher notes connecting graphing and algebra tiles ('A Geometry-Graphing Connection'):
    - ➢ http://tinyurl.com/GeomGraph
  - Partial product method for multi-digit multiplication (video; you will need to wait for a commercial to finish):
    - ➢ http://tinyurl.com/Partial-Prduct-Method
- The sections on manipulatives, base-10 blocks, and making a rectangle in the article 'Operation Sense, Tool-Based Pedagogy, Curricular Breadth: A Proposal' by Henri Picciotto, at:
  - ➢ http://tinyurl.com/Algebra-Lab-Gear-1
- Algebra tiles demonstration from Henri Picciotto's website:
  - ➢ http://tinyurl.com/Algebra-Lab-Gear-Demo
- Download and print out the following documents for Student Teacher use:
  - Rectangle graph activity ('Rectangles: Same Perimeter, Different Areas'):
    - ➢ http://tinyurl.com/RectangleGraph
  - 'Quick Guide' to various types of functions:
    - ➢ http://tinyurl.com/Functions-Quick-Guide
  - Graphing quadratic functions:
    - ➢ http://tinyurl.com/Graphing-Quad-Functions
  - Homemade algebra tiles:
    - ➢ http://tinyurl.com/HomeMade-Algebra-Tiles
  - Colour copy of 'Geometry-Graphing Connection' (including page one of the teacher notes):
    - ➢ http://tinyurl.com/Connect-Geom-Graph
- Bring to class:
  - Graph paper
  - Rulers
  - Scissors
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 starts with Student Teachers developing a table of values and a graph for what they probably anticipate is another linear function activity.

Unlike past sessions in which you were asked not to give Student Teachers guidelines about parameters and scaling, in this session you will need to be quite direct about this so that Student Teachers can focus on the attributes of the resulting graph without considering conflicting images.

Regarding their data table: it will have more than four columns, and you will need to tell Student Teachers which two columns to use when setting up their graph.

Once they have used their table of values to construct the graph, they may be in for a surprise: it is not a line. It's a parabola!

At this point you and the Student Teachers will do two things that will occupy the rest of the class period: 1) analyse their graphs and then 2) analyse their table of values.

When analysing their graphs, listen to the features that Student Teachers notice and be prepared to ask follow-up questions and comment about intersection with the x axis, a line of symmetry, the parabola's orientation (opening up or opening down), etc.

When analysing their table of values, Student Teachers will discover that what they learned about first differences for linear equations no longer holds true. What does a 'second difference' mean?

Student Teachers also should notice patterns in their tables with regard to symmetry, similar to this chart:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

It is important to ask a variety of questions about how their table relates to their graph.

A NOTE ABOUT LIMITED TIME: last week when slope was introduced, the focus was on positive slope because there was not enough class time available to discuss negative slope and negative y-intercepts. The same holds true here for quadratic graphs. There will be a homework assignment with a handout so that Student Teachers can see that some parabolas are upturned whereas others are downturned, some have a line of symmetry on the y-axis (as the above table implies), whereas others do not, etc. This is a good opportunity for them to use the free online graphing calculator to explore the eight different quadratic graphs on the homework sheet. This approach to using technology, in which Student Teachers spend time analysing eight computer-drawn graphs, may be a more valuable learning experience than having them draw each one. The homework sheet has equations in a format that allows Student Teachers to enter them into the online calculator and see not only the graph but also the corresponding table of values.

Session 2 continues the idea of multiple representations in algebra by adding quadratic expressions to the tables and graphs that Student Teachers produced in the prior session. Thus they will need to bring the prior session's work back to class to extend the work they did with tables and graphs.

Session 2 will show how the expanded format for a quadratic equation ($y = ax^2 + bx + c$) and the factored format are two different ways to express the same function.

The session will also relate these two formats to the table of values, graph, and algebra as generalized arithmetic. This will be an opportunity for Student Teachers to revisit the partial products method for multiplication that they studied in the 'Numbers and Operations' unit, noting how this method relates to quadratic equations.

A key aspect of this session will be the introduction of a manipulative called algebra tiles to model how the partial product method applies to quadratics, showing the link between the expanded and factored forms. It is imperative that you read through the 'Algebra Tiles' handouts and websites to prepare for this session.

Student Teachers have experienced *x* as a variable. In Session 3, the focus shifts to *x* as the unknown in a given equation and to symbol manipulation.

Phrases such as 'like terms' have not been specifically noted during this unit. However, by the time Student Teachers have reached the end of this unit they should have an intuitive grasp of them. When combined with their understanding of equations from the 'Numbers and Operations' unit, Student Teachers should be well positioned to use the commutative, associative, distributive, and identity properties to solve for *x*.

This final session of the unit will also ask Student Teachers to reflect on what they can do as teachers of young children in the early grades to foster the algebraic thinking that children will need in middle grades and secondary school.

Unit 2/week 4, session 1: Introduction to quadratics I – Tables and graphs

What do Student Teachers need to know?

Quadratic (square) equations can be modelled geometrically by rectangles.

A quadratic equation appears on a graph as a parabola.

Quadratic equations have a constant 'second difference' when seen in a table of values.

When placed on a coordinate graph, a parabola can be inspected for its maximum and minimum points, line of symmetry, and the places where it may cross the *x*-axis.

It is important to link drawings, tables, graphs, and symbolic ways of representing quadratic equations so that students have a comprehensive picture of how quadratic equations differ from linear ones.

How do children think about these concepts?

Children are often intrigued by a graph that shows a parabola.

This curved-shaped graph is something new. Because of this, children need to create (not just be shown) multiple representations of the same quadratic function to solidify their understanding of this new concept.

The point at which a parabola has a minimum or a maximum is called its *vertex*. This is a different interpretation of vertex than students know from geometry lessons in their earlier years. In their geometry lessons, the word *vertex* described a corner of a polygon or polyhedron.

When analysing their tables and graphs, children need to continually go back to their drawings. This will help them interpret how the maximum area of the rectangle is shown as the maximum point on the graph.

Numerically, when looking at their table they need to see that the maximum area of the rectangle was s^2 and that their drawing showed a square.

What is essential to do with Student Teachers?

- Have Student Teachers complete the sketches, table of values, and graph for a rectangle of varying dimensions but with a constant perimeter.
- Have a whole-class discussion about the characteristics of a parabola and the second differences in a table of values.
- Leave plenty of time to discuss the homework assignments, which will extend today's work by introducing second-degree equations and foreshadowing the next session's work with algebra tiles and the links between algebra and arithmetic.

Activities with Student Teachers

Begin by distributing the rectangle graph worksheet. Have Student Teachers work in pairs to sketch and label the dimensions of the various rectangles, then create a table of values and a graph to model the problem.

When Student Teachers have finished the assignment, begin by discussing the graph.

- How is it different from the graphs of linear functions that they worked with last week?
- Did the shape of the graph surprise them?
- How would they describe the graph's shape? If no one mentions the word *parabola*, introduce it now.
- Continue to probe about the parabola's features.
    - Does it cross the x-axis? If so, where?
    - What are the coordinates of those points?
    - How 'tall' is the parabola?
    - Which point on the parabola shows its maximum height?
    - Does the parabola appear symmetrical?
    - If so, could you draw its line of symmetry?
    - How could you express that?

After discussing the graph, discuss the table. Remind Student Teachers of the first difference they saw in the tables of values for linear equations. Have them write the first differences next to their table. Is there a first difference here? Is there another pattern? It is likely that some Student Teachers will notice a second difference. At this point, distribute the 'Quick Guide' to functions handout, which shows that degrees of difference can tell us about the shape of a function's graph.

Remind Student Teachers that they used three different representations today and that in the next class session they will be using two other representations, manipulatives and equations, to explore similar problems.

Also mention that the rectangle problem resulted in a parabola with specific characteristics. Not all parabolas look like this one. Their homework assignment will help show this.

Assignment

Distribute:

- 'Graphing Quadratic Functions' (first two pages of the PDF):
  - http://tinyurl.com/Graphing-Quad-Functions
- Homemade algebra tiles:
  - http://tinyurl.com/HomeMade-Algebra-Tiles

Give Student Teachers the URLs for:

- The free online graphing calculator:
  - http://tinyurl.com/Free-Graph-Calc
- The algebra tiles demonstration:
  - http://tinyurl.com/Algebra-Lab-Gear-Demo
- The video on the partial product method for multi-digit multiplication:
  - http://tinyurl.com/Partial-Prduct-Method

Have Student Teachers bring their algebra tiles and 'Graphing Quadratic Functions' assignment as well as today's work on rectangles to the next class session.

For the graphing quadratic functions assignment, Student Teachers will explore eight different quadratic equations expressed in the expanded form $y = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$, ideally by using the free online graphing calculator to draw the graphs.

Have them:

- Sketch the graphs on the worksheet.
- Analyse the graphs for patterns by looking at the coefficients of a, b, and c, and by making conjectures.

Have Student Teachers colour and cut out a set of algebra tiles to bring to the next class session.

The x and x^2 pieces should be coloured blue, and the '1' or unit pieces should be coloured yellow. These are the manipulatives they will use to model quadratic equations and show a link between arithmetic and algebra.

Have Student Teachers look at the algebra tiles demonstration to see how the tiles are used to model the factored and expanded forms of quadratic equations.

The third assignment is for Student Teachers to look at the video to review the partial product method of multi-digit multiplication (let Student Teachers know there will be an advert at the beginning for several seconds before the video continues).

Unit 2/week 4, session 2: Introduction to quadratics II – Equations, connecting algebra and arithmetic

What do Student Teachers need to know?

Quadratic functions can be represented by equations in the form of $y = ax^2 + bx + c$.

The coefficients a, b, and c in $y = ax^2 + bx + c$ determine the shape of the equation's parabolic graph.

The second differences on a table of values indicate a second-degree equation, where the greatest power of x is 2. (Similarly, a first difference indicates a first-degree, or linear, equation where the power of x is 1.)

A quadratic equation can be expressed either in its expanded form or its factored form.

Quadratic equations can be modelled by a manipulative called algebra tiles.

The partial products method of multi-digit multiplication, introduced in the 'Numbers and Operations' unit, can be directly linked to algebra tiles.

How do children think about these concepts?

Because numbers and operations are the major focus of early education, children need strategies to help them link their number sense to algebra concepts.

Children who are visual learners need ways to link algebra to their geometric sense.

Children who are tactile learners need to work with manipulative materials, moving them around in space, to make sense of mathematical concepts.

All these different kinds of learners can benefit from using algebra tiles to connect algebra to what they already know as well as how they need to learn it.

When presented with various representations of a quadratic function, children need their teacher to make connections among various representations.

For example, children may not notice that the maximum or minimum of a graph is evident from the table of values. Or that the c in the expanded form of $y = ax^2 + bx + c$ is the y-intercept.

Rather than telling children about these characteristics, teachers need to ask probing questions that force their students to look for patterns. This means that teachers must find specific examples to elicit children's thinking about quadratic patterns.

What is essential to do with Student Teachers?

- Have a whole-class discussion of the 'Graphing Quadratic Functions' homework, exploring the relationship between a quadratic equation in its expanded $y = ax^2 + bx + c$ form and the shape of its graph.
- Introduce algebra tiles as a way to model quadratic equations in factored form.
- Link the algebra tile model (using x as a variable) to the partial product multiplication model used in arithmetic.

Activities with Student Teachers

Begin by reviewing the homework assignment, 'Graphing Quadratic Functions'.

Student Teachers were asked to draw the graph of equations by using a graphing calculator, sketch the graphs, and then make conjectures about the connections between a graph's shape and the coefficients of the terms in its equation.

Begin reviewing the homework by reminding students of the generalized form for a quadratic equation: $ax^2 + bx^1 + cx^0$. (This helps clarify that c, which looks as if it has no variable, really has x to the 0-power. A discussion on this point is not warranted now, but it does show Student Teachers that there is a pattern to the exponents of x in the equation.)

Then ask questions such as:

- What happened when the coefficient a was negative?
- Will the parabola open up or down?
- Did any of the equations shift to the right or left of (0,0)?
- Which coefficient seemed to cause that?
- Which coefficient made a graph wider or narrower?
- What does the coefficient c show?
- Did you find any pattern for the coefficient b?

Student Teachers may have been confused by the factored form in the last two equations, as last week's lesson on order of operations did not mention exponents. This is an opportunity to ask them how they interpreted $2(x - 4)^2$. Mention that the factored form will become clear once they have worked with their algebra tiles.

Review the partial product method for multi-digit multiplication by having Student Teachers decompose 23 x 17 on to graph paper: 20 units + 3 units horizontally, and 10 units + 7 units vertically. What does their filled-in grid look like? (200 units + 30 units + 140 units + 21 units = 391 units.) Now ask them to draw a quick diagram on blank notebook paper with just the numbers 20 and 3 horizontally and 10 and 7 vertically.

Can they quickly multiply those numbers to fill in the grid? What numbers did they come up with? Ask if it mattered in finding the answer that they used actual units versus numbers once they understood the process.

Remind Student Teachers that when they decomposed the numbers 23 into (20 + 3) and 17 into (10 + 7), they were creating factors that will help them think about quadratic equations when they use algebra tiles.

Have Student Teachers work in pairs for this activity. Introduce the algebra tiles by having Student Teachers model the expression $(2x + 3)$ x $(x + 7)$ by arranging on their desktop two x pieces and three 'unit' pieces horizontally, and one x piece and seven 'unit' pieces vertically.

Next, have them fill in the products with their x^2, x, and unit tiles. Discuss:

- What did they notice?
- How does this algebraic result relate to the work they just did with numbers?
- How does the factored form relate to the total number of algebra tiles (the expanded form)?

Next, have Student Teachers use their algebra tiles to create a rectangle that contains two x^2 tiles, five x tiles, and three unit tiles. (This is a model for $2x^2 + 5x + 3$.)

What are the dimensions of this rectangle? What are the factors of $2x^2 + 5x + 3$? (This task, working backward from the expanded form to its factors, is more challenging than the previous task.)

If time allows, have Student Teachers solve $y = x^2 + 3x + 4$, substituting 10 for x. Then solve the following equations, substituting 10 for the value for x:

$$2x^2 + 3x + 4$$
$$5x^2 – 3x – 4$$
$$(x + 8) * (x + 2)$$
$$(x – 3) * (x + 5)$$

How do the above expressions relate to the partial products method for two-digit multiplication?

Distribute colour copies of the homework assignment, 'A Geometry-Graphing Connection'.

Assignment

Ask Student Teachers to complete the colour copy of 'A Geometry-Graphing Connection' (three pages; include teacher notes):

➢ http://tinyurl.com/connect-geom-graph

Unit 2/week 4, session 3: Solving for *x*, the unknown

What do Student Teachers need to know?

Variables are letters that represent unknown numbers or values.

The concept of using x to represent a specific number or numbers (such as in the equations $3 + x = 5$ or $x = \sqrt{4}$) differs from the concept of using x to represent any number or value in a function (such as in $y = 3x + 5$).

The properties of arithmetic (commutative, associative, distributive, and identity) also hold true for (and can be applied to) algebraic equations.

Solving an equation for x relates to the 'balance model' of equivalency.

To create this balance, students need to use 1) symbol manipulation, 2) arithmetic properties, and 3) order of operations.

On the function's table of values, any given value for x has a corresponding y value.

How do children think about these concepts?

In arithmetic, equations are usually written so that the answer comes on the right-hand side of the equation, with the typical syntax of a number sentence being '4 plus 3 equals 7'.

In algebra's use of symbolic notation, that syntax is often reversed so that the answer is on the left of the equals sign, which would result in the arithmetic equation '$7 = 4 + 3$'.

After many years of working with equations in the format of $a + b = c$, students can find it difficult to shift to a syntax where the answer is on the left-hand side of the equation and where the expressions and operation signs are on the right-hand side, such as in $7 = 4 + 3$ (or its corresponding algebraic equation, $7 = 2x + 3$).

When working with linear equations in the format $y = mx + b$, children may not understand that the dependent variable y can have a coefficient. For example, in the equation $4y = 2x$, children may not realize they need to divide both sides of the equation by 4 in order to maintain equivalence. Instead, they may interpret the equation as either $y = 2x + 4$, or $4y = 4(2x)$.

Even if children know that variables can have a coefficient, they may not realize that a coefficient does not need to be a whole number. They need to understand that coefficients can be any type of number they have studied: integers, fractions, or decimals.

Children may assume that once they have found the value for x in a given equation, that x has that same specific numerical value for x in other equations. For example, suppose there are three problems on a worksheet, the first being $x + 3 = 7$. The teacher expects that a child will correctly solve for x and discover that its value is 4. If the next problem is $x + 5 = 9$, the value for x is still 4. However, if the third problem is $x + 3 = 9$, children may be looking for a pattern and determine that x must once again be 4.

What is essential to do with Student Teachers?

- Review the algebra tiles homework assignment using prompts from the teacher notes section.
- Introduce a new definition for x: x, the unknown. Note common misconceptions that children may have about this.
- Review the balance model of equivalence and how it works in algebra.
- Review the commutative, associative, distributive, and identity properties and note how they work in algebra.
- Introduce the idea of 'like terms' and how they can be combined to solve for x.
- Note the strategy of isolating x by using the arithmetic properties and the concept of equivalence.
- To end this algebra unit, allow time for a discussion about Student Teachers' reflections on what they think about algebraic thinking and the teaching of algebra—which may be quite different from how they learned this subject when they were in school.

Activities with Student Teachers

Begin by reviewing the homework assignment, 'A Geometry-Graphing Connection'. How did the questions about the algebra tiles as a visual model relate to the past assignment 'Graphing Quadratic Functions'? What did Student Teachers notice about the equations of 'incomplete rectangles' using the algebra tiles? What does that imply?

Use the discussion prompts in the section on parabolas. (Because you have given the teacher notes page to Student Teachers, they should be prepared to discuss these questions.)

Shift the focus to x as the unknown in a given equation.

Have Student Teachers quickly create a table of values for the linear function $2x + 3$, beginning with $x = 0$ and ending with $x = 5$. Ask them how they would use their table to evaluate the expression when $x = 2$. Did their answer reflect the y-value for x as 2?

Ask Student Teachers how they learned to evaluate expressions and solve equations when they were at school. How is this table-of-values method different from the method of substituting 2 for x in the expression $2x + 3$?

Have Student Teachers think about what they learned about equivalence in the 'Numbers and Operations' unit. How does the balance model help them understand algebraic equations across the equals sign? How can they use what they've learned about properties, order of operations, and equivalence to isolate x and find the unknown?

End the session by having Student Teachers respond to these prompts for their reflection, then have a whole-class discussion about their thoughts.

- What mathematical or pedagogical ideas from the 'Algebra' unit stand out as I think about my future students?
- What steps would I take to prepare my students to be algebraic thinkers ready for their future algebra courses?

Assignment

To be determined by the Instructor.

UNIT 3

GEOMETRY AND GEOMETRIC MEASUREMENT

FACULTY NOTES

Unit 3/week 1: Pre-assessment, polygons, similarity, benchmark angles

Session 1: Unit pre-assessment
Session 2: Characteristics of polygons, regular and irregular polygons, classifying polygons, hierarchy for polygons
Session 3: Introduction to similarity, benchmark angles

Faculty preparation for the upcoming week (1–2 hours)

- Look through the following webpages that address polygons:
  - Hierarchy of polygons:
    - http://tinyurl.com/Polygon-Hierarchy
  - Sorting polygons:
    - http://tinyurl.com/Sorting-Polygons-1
  - Venn diagram about quadrilaterals:
    - http://tinyurl.com/Quads-Venn
- Download and print out for Student Teacher use:
  - 'Attributes of Polygons', which includes an image of 25 polygons (available as a resource in the Course Guide)
- Bring to class:
  - Lined paper
  - Plain paper
  - Graph paper
  - Rulers
  - Scissors
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 begins with a pre-assessment that helps the Instructor discover Student Teachers' current understanding of geometry topics that are included in the elementary curriculum.

Session 2, the first instructional session of the unit, begins with polygons. Many texts begin with angles (a more abstract concept) and then move to polygons (which are more recognizable in the environment).

As with Unit 1, which focused on numbers and operations by moving from the concrete to the abstract, we will begin by exploring polygons so that Student Teachers will begin to notice the role of angles in forming these shapes.

This method, based on research that helps teachers understand how children think about geometry, is a helpful model in a university course supporting Student Teachers.

During this session, Student Teachers will learn to recognize the characteristics of polygons and begin to classify them according to number of sides, side lengths, and eventually equal versus non-equal angles.

Session 3 will address similar polygons and use a 90-degree right angle as a benchmark from which other angles can be devolved and to which other angles can be compared.

Unit 3/week 1, session 1: Unit pre-assessment

What do Student Teachers need to know?

This first day of the 'Geometry' unit is focused on assessment. It is an opportunity for the Instructor to discover what Student Teachers know, think, and remember about geometry.

It addresses (and pre-assesses) the vocabulary, concepts, and skills that Student Teachers will study during the next five weeks:

- Vocabulary and terminology
- Polygons and circles
- Angles
- Cuboids and cylinders
- Area, volume, and surface area
- Right triangles and the Pythagorean theorem

It is important to take time to assess Student Teachers' prior knowledge, as it is likely that most of them equate geometry with their secondary school coursework rather than thinking of geometry as part of their everyday lives: wrapping a package (surface area), taking a shorter diagonal route through a park (Pythagorean theorem), or holding a football (a sphere covered with pentagons and hexagons).

Similarly, if Student Teachers think of geometry as a secondary school subject dealing with proofs and formulae, they may not realize how accessible geometry is for young children.

The questions suggested in the pre-assessment below are not meant to be scored. Rather, they offer a window on Student Teacher's thinking that will be addressed in subsequent sessions over the next five weeks.

What is essential to do with Student Teachers?

- The following whole-class discussion should be lively and engaging, with a focus on 1) discerning what Student Teachers already know and 2) stimulating Student Teachers' thinking by asking probing questions and providing clues rather than giving answers.
- As you pose the following types of questions, note the vocabulary and terminology that Student Teachers use when describing their thoughts. Is their terminology formal or informal? Consider how the phrasing of your questions can model for Student Teachers how to become more formal and precise in their geometric language.

- Geometry is visual mathematics. Thus, during the discussion ask Student Teachers to come to the board to draw what they are trying to communicate. Student Teachers should also be drawing and labelling diagrams in their notebooks and noting points of confusion that can be followed up in later sessions during the unit.
- Because this class meeting is a pre-assessment, try not to go in depth when addressing individuals' misconceptions. There will be ample time and opportunities to address confusion during the remainder of the unit.

Activities with Student Teachers

Here are some sample prompts that are designed to help Student Teachers think about the many concepts they will encounter during the 'Geometry' unit.

'When did you first start learning geometry?' If Student Teachers say they first learned about geometry in secondary school, ask about what they think young children might already know about geometry.

'Which geometric shapes and forms do you see in this room? Do you think young children would be able to notice those things, too? How could you help children begin to see the geometry in their environment?' (For example, the shape of the windows, tiles on the floor, stripes on clothing, or the circle of a clock.)

Extend questions about these two-dimensional shapes by asking about three-dimensional forms, such as a book (cuboid), pencil (cylinder or hexagonal prism), ball (sphere), or cardboard packet (cuboid).

After this informal discussion about polygons and polyhedra, ask about the generic names for polygons, which are dependent on the number of a polygon's sides: *triangle, quadrilateral, pentagon, hexagon* (you may omit the seven-sided septagon or heptagon), and *octagon*.

Ask why there are no two-sided polygons.

Continue by asking, 'What do you think is implied by the term *regular* polygon? Why might a given polygon be called *regular* versus *irregular*?'

When talking about polygon shapes, do Student Teachers mention parallelograms, rhombuses (rhombi), and trapezoids?

Do they realize that a square is a rectangle? If so, can they explain why? Is a square a rhombus? If so, can they explain why?

Ask about circles. Circles are closed shapes but not polygons. This might give rise to a definition of a polygon as well as descriptions of shapes that are not polygons.

Is a ball a circle? If so, why? If not, why not? What about a cube versus a square?

Is a box a two-dimensional shape (such as a box to tick on a survey) or a three-dimensional form (such as a packet of rice)?

Why do you think young children (and even adults) use these kinds of terms incorrectly and interchangeably?

Have Student Teachers consider a square and a cube.

What is a side? What is an edge? A face? A corner? A vertex?

Which terms apply to two-dimensional shapes? Which ones apply to three-dimensional forms? How do we use these terms informally? (For example, ingredients are listed on the *side* of the food packet. But that side is really the *face* of a cuboid when we consider that same packet in geometric terms.)

Once Student Teachers have considered these concrete aspects of geometry, move to more abstract, undefined terms, asking them to describe their thinking about the following words: *point* versus *dot*, *line* versus *line segment*, *ray*, *degrees*, *plane*, *parallel*, and *perpendicular*.

When addressing angles, ask which angle comes to mind first. (Most likely it will be a 90-degree right angle.) Ask about other angles that they could derive from a 90-degree angle. Do they mention a 180-degree straight angle? A 360-degree angle?

Sketch (no need to construct) different angles on the board. Ask Student Teachers to describe them. What terminology do they use?

Move to geometric measurement and ask what Student Teachers remember learning about area and volume. Do they respond with formulae? If so, ask how young children (who don't know formulae) might be thinking about area. How might these same young children think about volume? What about surface area? What kind of real-life examples do Student Teachers give when discussing geometric measurement?

Finally, ask about measurement as applied to right triangles. What do Student Teachers recall learning about right triangles? Do they offer the formula for the Pythagorean theorem? Ask how the formula might be proved. (Which is very different from simply recalling the formula.) Ask where the Pythagorean theorem might be used in everyday life.

At this point you have introduced all the major topics to be addressed during this five-week unit.

You also have an overall picture of the Student Teachers' understanding about these topics. Are they wedded to formulae? Do they see geometry around them? Do they consider how young children can approach geometry?

Assignment

To be determined by the Instructor.

Unit 3/week 1, session 2: Characteristics of polygons, regular and irregular polygons, classifying polygons, a hierarchy for polygons

What do Student Teachers need to know?

Polygons are closed two-dimensional shapes formed by line segments that meet at a corner, or *vertex*.

A polygon is named by the number of its sides.

If all sides of a polygon are equal, their interior angles are equal, and thus the shape is termed a *regular polygon*.

The shape of a polygon depends not only on its side lengths but also on the measures of its interior angles.

Polygons can be classified according to their side length and angle attributes.

How do children think about these concepts?

When considering triangles, most young children think of an equilateral triangle standing on its base. If the same triangle is rotated so that one of its vertices is pointed downward, they often do not see it as a triangle.

Similarly, if shown a scalene triangle, with three different side lengths, they also might not perceive it as a triangle. This is because they have equated their concept of a triangle with the equilateral model. They have not yet discovered that a triangle is any three-sided closed figure.

Quadrilaterals can be organized into a hierarchy, from an irregular quadrilateral to the most specialized quadrilateral, a square.

Most children do not realize that a square is also both a rectangle and a rhombus, as well as a parallelogram. Just as with triangles, they may believe this because they have not had experience discussing the characteristics of quadrilaterals.

However, they also lack the vocabulary and terminology (such as *parallel* and *opposite*) as well as the idea of angle measurement. Thus, once they realize a polygon has four equal sides but doesn't look square, they may think of it as a 'squashed square', where two of the vertices look pointier than the other two. This informal description of a rhombus is a starting point from which the teacher can begin to formalize how to describe the shape's characteristics.

Children may assume that any closed figure is a polygon or that an open figure composed of line segments is a polygon. Thus it is important to offer counterexamples—such as those below—so that children can begin to refine their definition of a polygon's required characteristics.

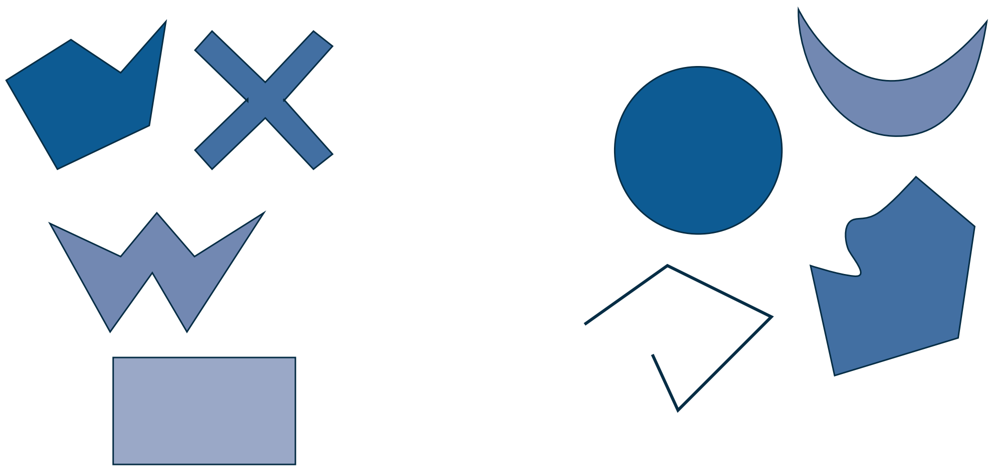
What is essential to do with Student Teachers?

- Introduce polygons and several counterexamples, having Student Teachers develop a working definition for what constitutes a polygon.
- Introduce regular and irregular polygons, having Student Teachers develop a working definition (including both side length and angle measure) for what constitutes a regular polygon.
- Have Student Teachers classify polygons according to their attributes.

Activities with Student Teachers

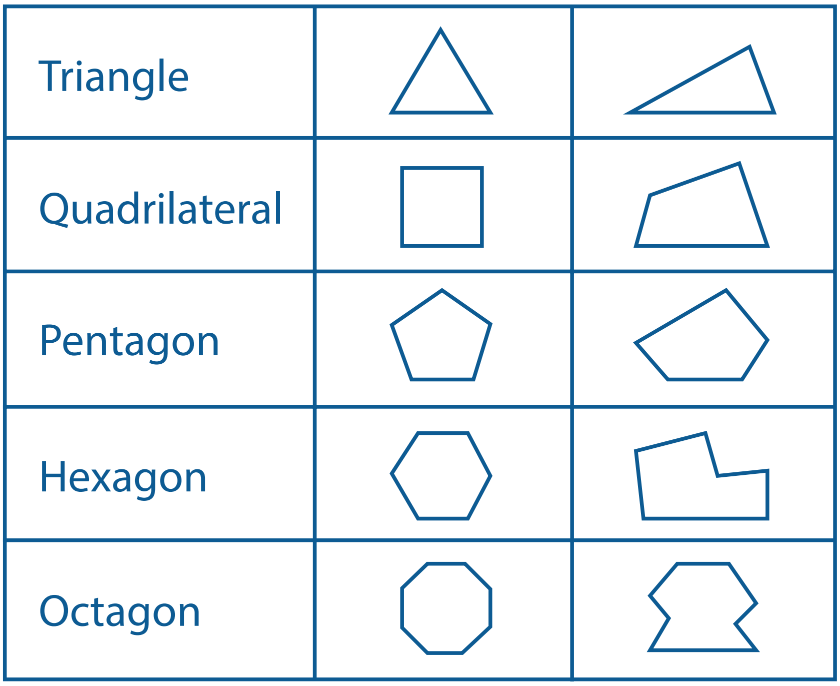
Begin by referring to the comments about polygons Student Teachers made in the previous session, noting that those ideas will become more precise during today's class.

On the board, draw figures of polygons and counterexamples, such as the ones in the picture below. Ask for thoughts as to why you divided the shapes into two columns:



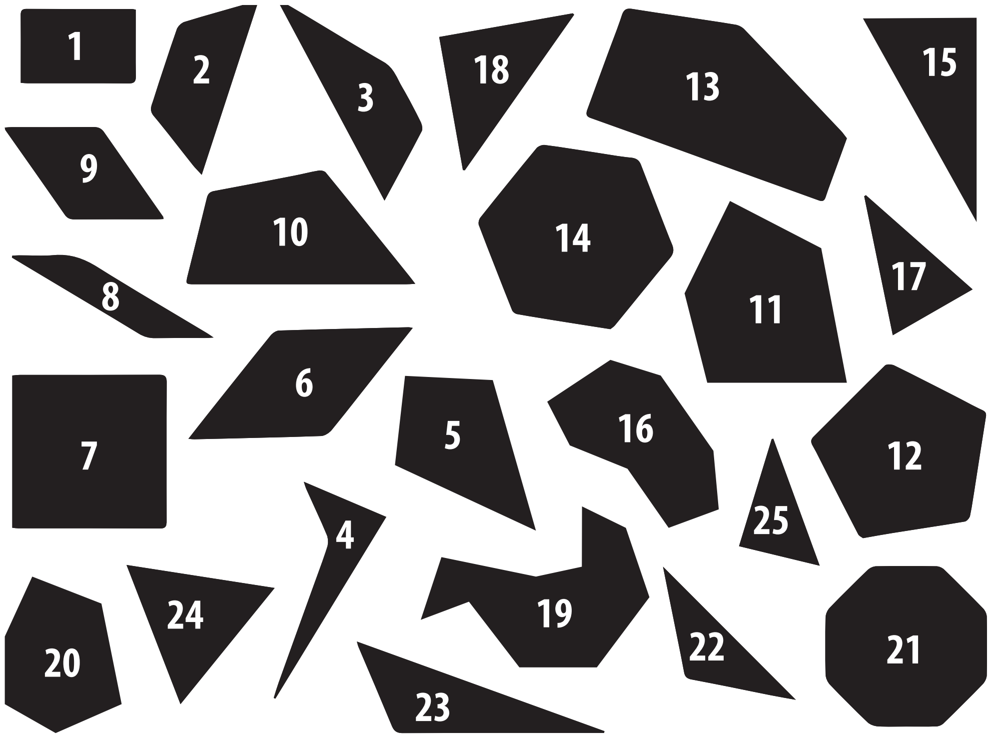
As Student Teachers talk about their ideas, press for greater precision in the way they describe the different shapes to come up with a working definition of what is and is not a polygon. Note that they may not think of concave polygons in the same way they think of convex polygons. Or that may call the rectangle a regular polygon, when the word *regular* has a special mathematical meaning that will be explored in the next activity.

Because all the polygons in the picture above are irregular polygons, introduce the idea of regular polygons by drawing on the board another two-column chart similar to that below.



Again, listen to the way Student Teachers attempt to describe the differences between the two columns. Do they note not only the side lengths but also the angles? If they say, for example, that the triangle on the left is equilateral, ask what they mean by that word. Is a square *equilateral*, too? What about the word equiangular? What might that mean? And how does it apply to the two pentagons? By the end of this discussion, Student Teachers should have come up with a working definition of regular and irregular polygons. Finally, refer back to the rectangle in the picture above. Ask why a rectangle, that is not a square, is not a regular polygon. Does their working definition help clarify this?

Distribute the 'Attributes of Polygons' worksheet.



Divide the Student Teachers into five groups; give each group one of the sets of polygon characteristics in order to classify the 25 shapes on the handout. Note that several polygons may fit the same category or that a particular polygon may fit into several categories. Notice the term *congruent*, which you may need to define for Student Teachers if they haven't used the word already.

Have a spokesperson for each group report the results of their particular discussion. Have Student Teachers note omissions and areas of confusion in each others' presentations, justifying their responses.

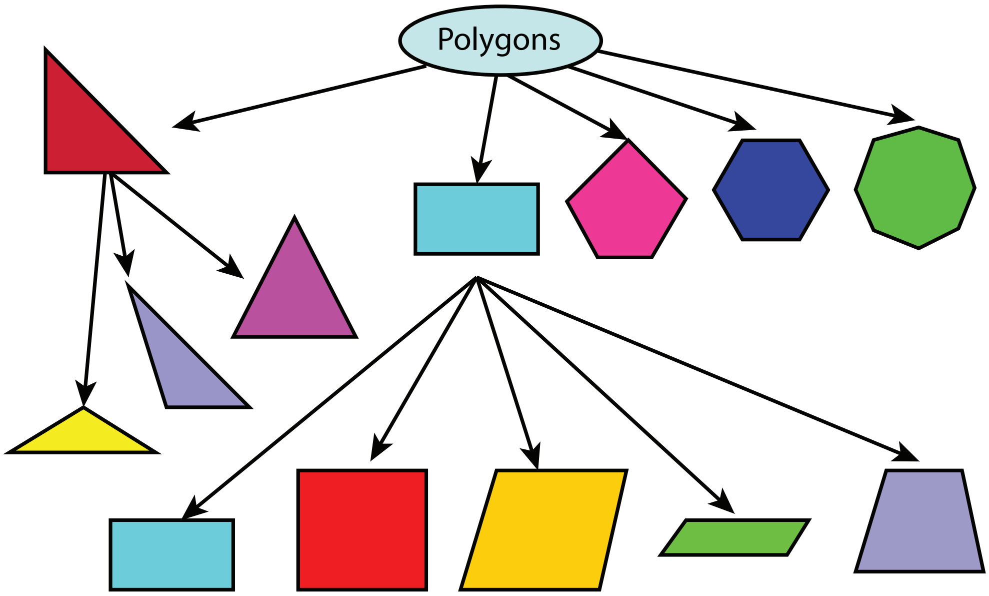
End the session by reviewing and acknowledging some of the more formal terminology Student Teachers had begun using as the session progressed.

Assignment

Have Student Teachers bring their 'Attributes of Polygons' worksheet to class for the next session, Session 3.

Have Student Teachers visit this website for visual models that show how various polygons are related:

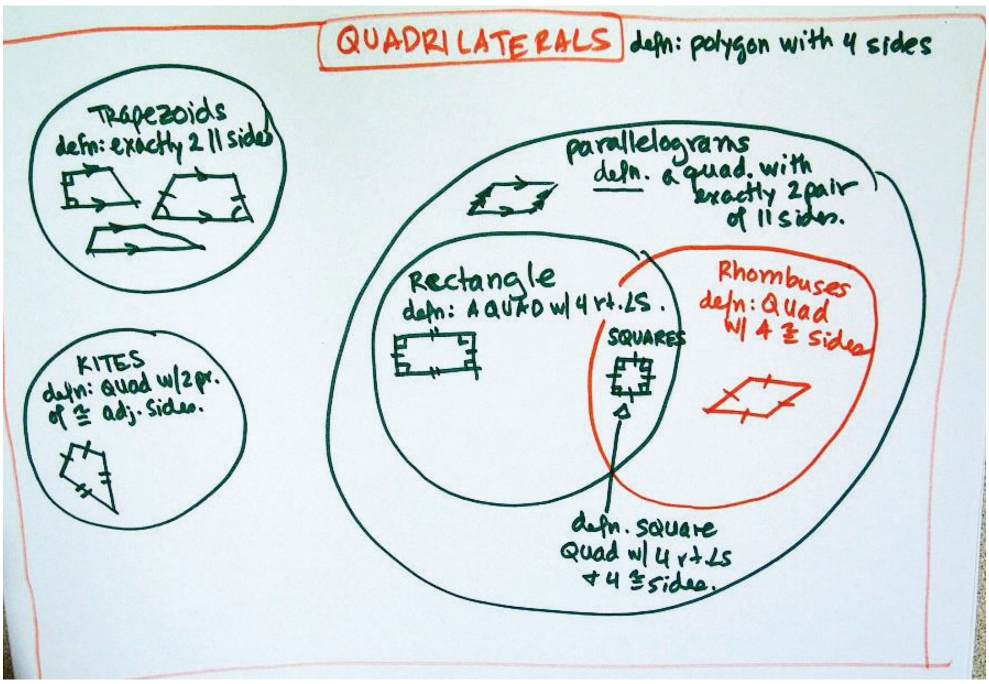
➢ http://tinyurl.com/Polygon-Hierarchy



Label the shapes using the following words: triangle, quadrilateral, pentagon, hexagon, octagon, square, rectangle, parallelogram, rhombus, trapezoid.

Quadrilateral Venn diagram:

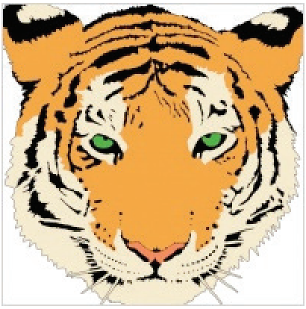
➢ http://tinyurl.com/Quads-Venn

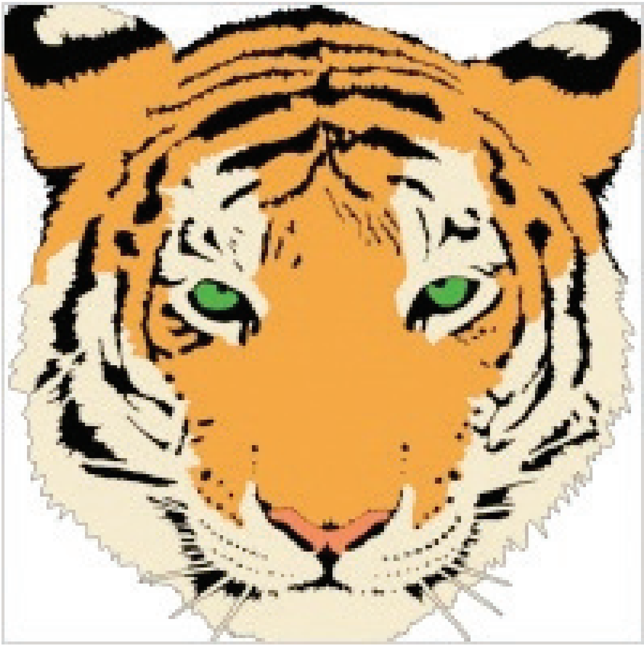


Unit 3/week 1, session 3: Introduction to similarity, benchmark angles

What do Student Teachers need to know?

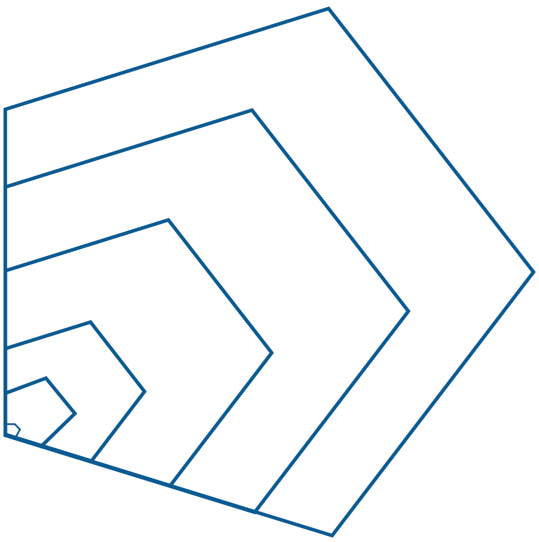
Similar figures are those whose side lengths are in direct proportion to each other, with the corresponding angles equal in degrees. Consider this drawing of a tiger:





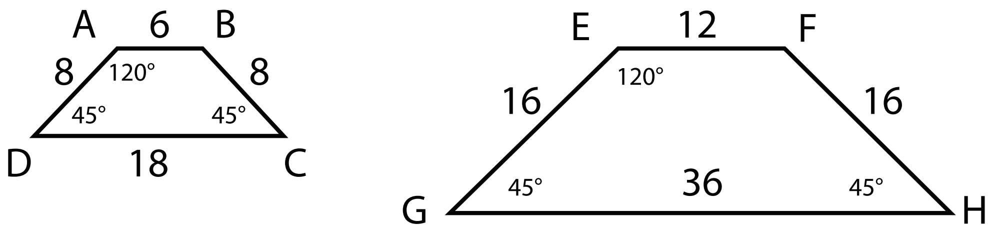
The larger drawing resulted by using a scale factor of two, doubling both the length and width of the original picture.

When dealing with geometric shapes (which are far simpler than the drawing of the tiger), each side of the figure is multiplied by the same scale factor while the corresponding angles remain the same. Look at this example of a regular pentagon that retains its shape regardless of the scale factor used:



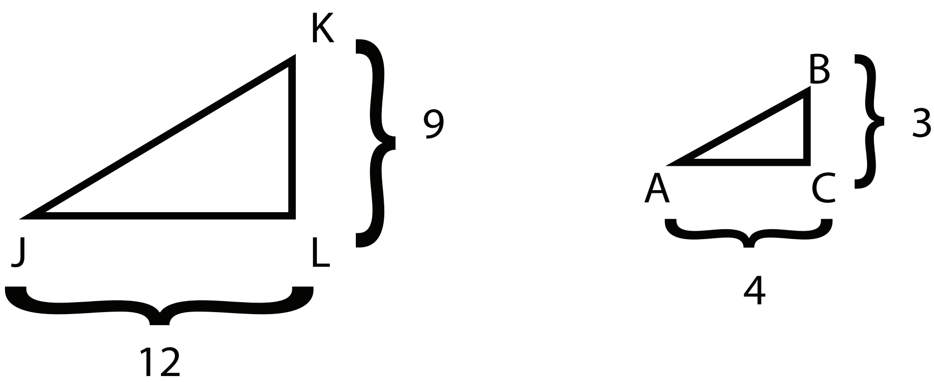
Scale factors are just that: factors. As such, the operations used when creating similar figures are multiplication and division (or multiplying by a scale factor less than 1).

The mathematical sign to indicate that two shapes are similar is '~'. Thus, the final sentence for the first set of diagrams would be read, 'Figure ABCD is similar to figure EFGH'.



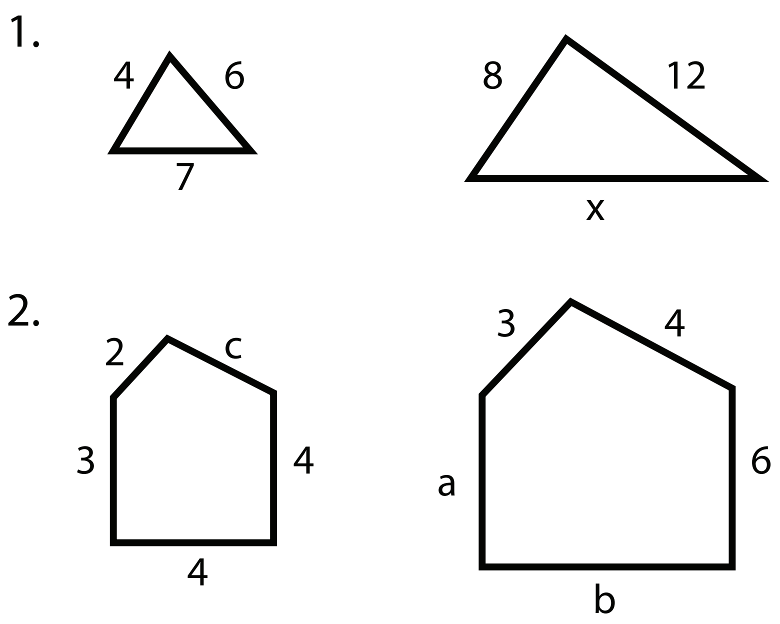
Since: <A = <E, <B = <F, <D = <G, <C = <H,
AB/EF = AD/EG = BC/FH = DC/GH; so ABCD ~ EFGH

Δ JKL ~ Δ ABC
The similarity ratio is 3/1



Knowing that corresponding side lengths are in proportion to each other can help solve for missing information. Consider these two sets of figures with missing information. How can the relationship between the side lengths of 6 and 12 help solve for *x*?

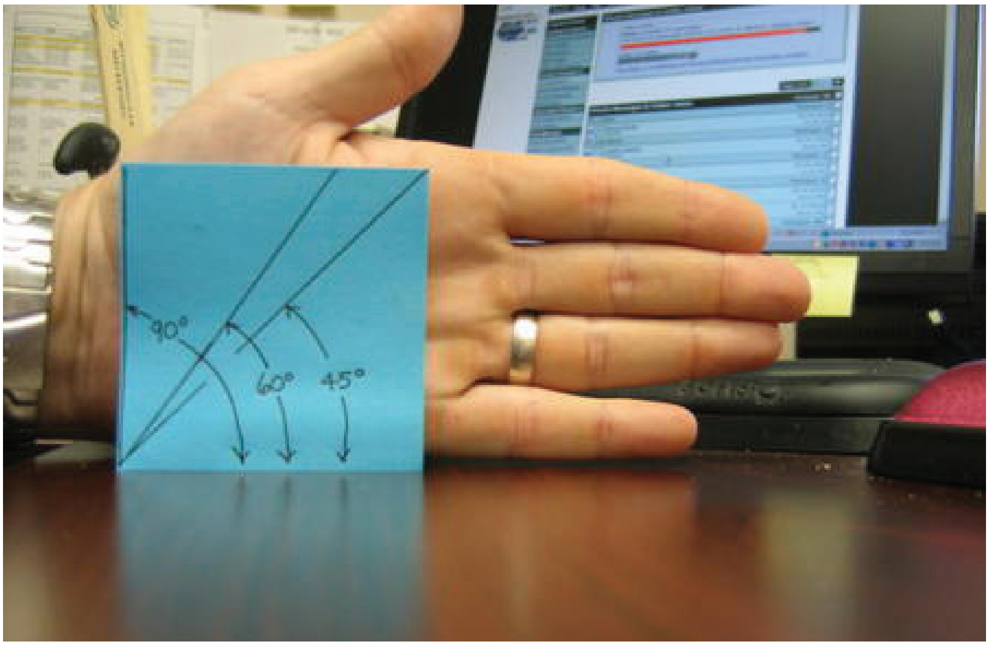
A somewhat more difficult situation is involved in the second set of pictures. What is the scale factor here?



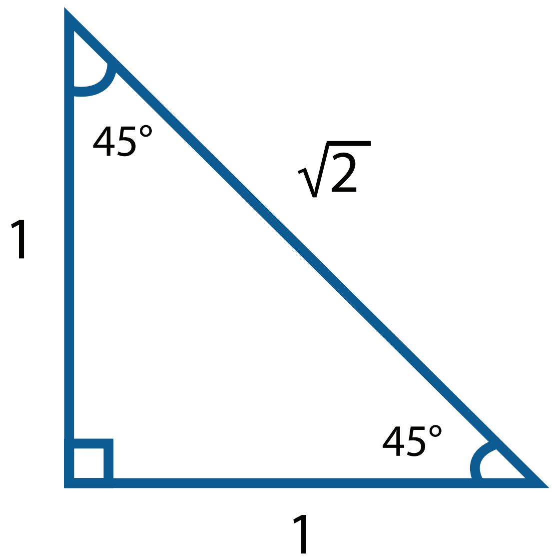
Finally, for an amusing real-life example of similarity:

Benchmark angles are those that serve as a reference point. They can be used both to devolve other angles and to compare angles.

The most common benchmark angle is the 90-degree right angle. Using the 90-degree angle as a reference point, one can devolve the 180-degree straight angle and the 360-degree revolution angle, and estimate the 45-degree, 22.5-degree, 135-degree, 60-degree, 30-degree, and 120-degree angles.



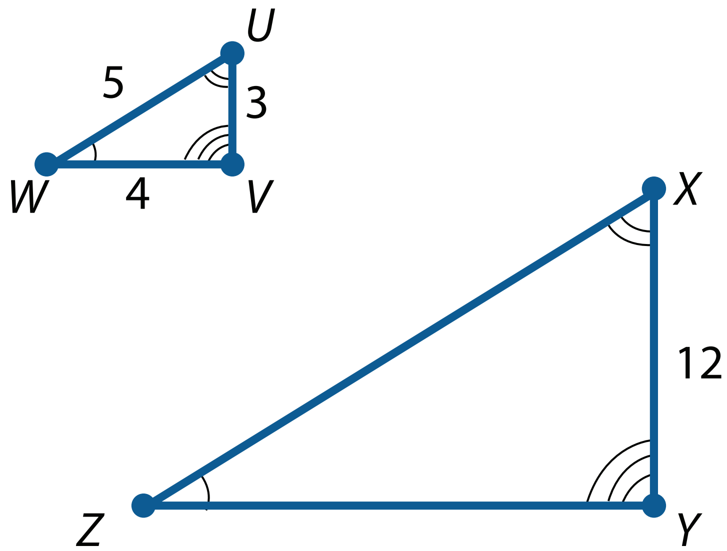
Note that certain polygons include these specific angles. For example, an equilateral triangle has three 60-degree angles for a total of 180 degrees. A square, with four 90-degree angles, has an interior angle sum of 360 degrees. If a square is cut on the diagonal, it will have one 90-degree angle and two 45-degree angles. (Again, each of the resulting two triangles has an angle sum of 180 degrees.)



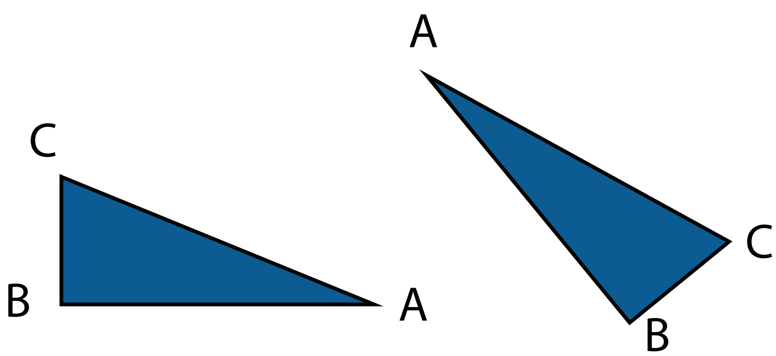
Other benchmark angles devolved from the 90-degree angle can be used for comparing angles and estimating their measurement. For example, if one knows how to create a 45-degree angle, one can determine if a given angle is greater or less than 45 degrees. The same would be true for an angle that looks to be between 90 degrees and 135 degrees.

How do children think about these concepts?

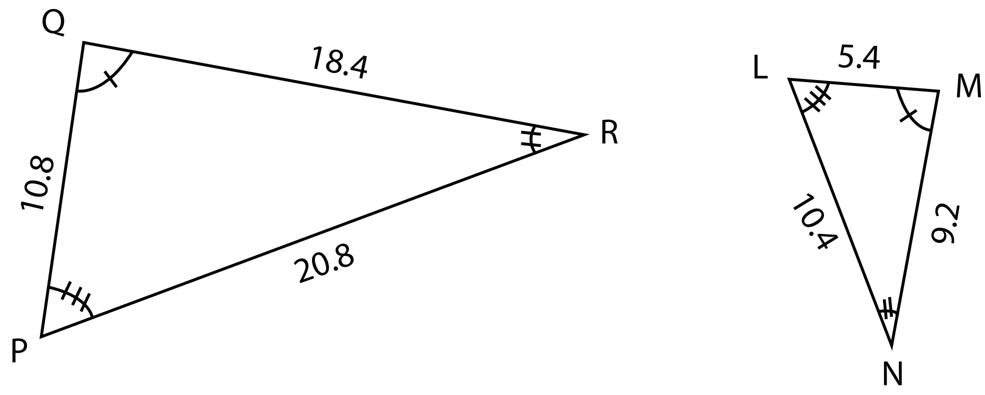
One of the most important considerations when trying to establish similarity between two figures is their corresponding sides. This is relatively easy if the figures have the same orientation as in this diagram (which also allows for establishing the length of other sides based on the 3:12 ratio of the vertical side).



But if the orientation of the shapes is not aligned, as in the figure below, children may fail to see which are the corresponding sides. In this picture, the two triangles are exactly the same (congruent), but the orientation is different.



In this second set of similar triangles, both the orientation and the scale factor are different. In testing for similarity, it is helpful for children to rotate their worksheets and notebooks or use thin paper for tracing one figure and placing it over the other.



(This is much like a younger child not realizing that an equilateral triangle with its base on the horizontal is congruent to the same triangle with a vertex 'down'.)

Children (and Student Teachers) may not understand that to create similar figures, a scale factor involves multiplication or division, not addition or subtraction. Here's an extract from a mathematics journal in which the writer describes this challenge:

> For example, suppose a triangle has side lengths of 2 cm, 4 cm, and 7 cm. Multiplying these side lengths by 2 produces 4 cm, 8 cm, and 14 cm. Therefore, a triangle with these side lengths would be similar to the original. However, adding 2 to each original side length will produce a triangle that is 4 cm, 6 cm, and 9 cm. It is not similar to the original, although students often mistakenly think that it is. … After several semesters of attempting to explain, in abstract terms, why multiplication and division are necessary where similar figures are concerned, I realized that many pre-service teachers need visual and hands-on experiences.
>
> (G. Johnson, '*Mathematical Explorations: Similar Triangles*', Mathematics Teaching in the Middle School, 16:4 (2010), 248–254.)

Even when children have had experiences that prove that the degrees in the corresponding interior angles of two similar figures are equal, they often hold on to the mistaken belief that a larger figure (with its longer sides) should also have more degrees in its angle sum than the smaller figure.

The fact that side lengths can vary (in proportion to the original) but that the measurement in degrees of the corresponding angles remains the same is much like the idea (to be discussed later) that the length of an angle's rays is irrelevant to the angle between those two rays.

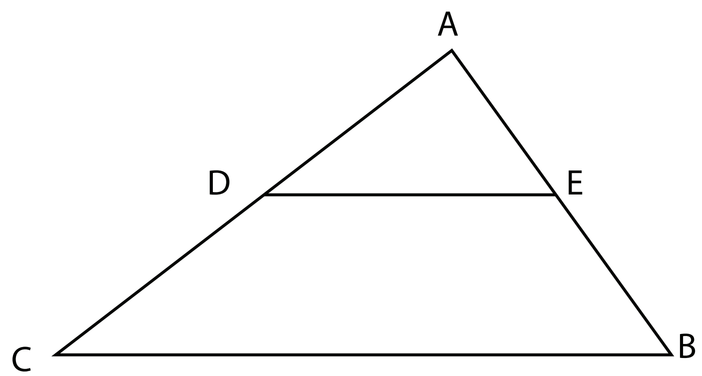
What is essential to do with Student Teachers?

- Corresponding sides and corresponding angles are the key elements of similarity. Corresponding sides must be in proportion; corresponding angles must have the same measure in degrees.
- Scale factors to create similar shapes use the operations of multiplication and division, not addition and subtraction.
- Using a 90-degree benchmark angle allows for a relatively accurate estimation and comparison of angles. This informal way of estimating angle measurements is a precursor to greater accuracy when measuring and constructing angles using a protractor.

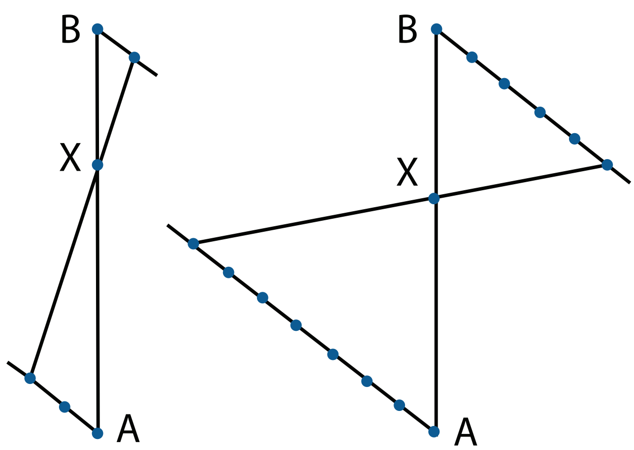
Activities with Student Teachers

Begin by reviewing any thoughts Student Teachers voiced during the pre-assessment about similarity and angle measurement. Let Student Teachers know that they will be exploring the concept of similar figures in several drawing activities.

Have Student Teachers take a sheet of lined paper and draw a large triangle, with the vertex on the top line and its base on the bottom line. Have Student Teachers draw a horizontal line parallel to the base from one side of the triangle to the other. What do they notice about the new triangle that they created inside the original one?



On a sheet of plain paper, have Student Teachers draw two 20-centimetre intersecting lines (a large 'x'). It is not necessary that the lines intersect at their halfway points. Rather, it is preferable for the understanding of similar triangles that the lines cross at a point other than the halfway point, such as:



Once Student Teachers have done this, have them join the endpoints. What do they notice about the two triangles they have created?

Next, have them fold the paper on the point of intersection and hold it up to the light. The two triangles will be overlapping. What do they notice now? Why is this so?

As you listen to their explanations, help Student Teachers move toward mathematically precise terminology:

- The *vertical angles* (a new term introduced here) created by intersecting lines are equal.
- The principle of corresponding-side-angle-corresponding-side is a way of confirming similarity.

(These are formal, secondary school ways of discussing similarity. However, Student Teachers should be able to justify similarity in this way even though they will describe it in simpler terms for children.)

Make sure Student Teachers realize that a similar shape's corresponding angles are the same even though its corresponding sides differ in length. The next activity will focus on how the proportionality of side lengths is a key factor in similarity.

On a sheet of graph paper, have Student Teachers draw three rectangles:

1) 1 unit x 2 units
2) 5 units x 6 units
3) 4 units x 8 units

Do they think these three rectangles look similar? (Each has four 90-degree angles, so the angle principle of similarity holds true.)

What do Student Teachers notice as they compare the side lengths of these rectangles? Do Student Teachers notice differences in proportion that occurred when adding 4 to both dimensions of the original 1 x 2 rectangle? Can addition of equal units to corresponding sides justify similarity ($1 + 4 = 5$, and $2 + 4 = 6$)?

What happened when the original 1 x 2 rectangle had each of its side lengths multiplied by 4, resulting in the 4 x 8 rectangle? Are rectangles 1 and 3 similar? If so, why? If not, why not?

Summarize this section of today's lesson on similarity by asking the following questions:

- How does the side length property relate to the term *scale factor*?
- How can knowing the scale factor help solve for missing sides on two similar figures?
- In the rectangle problem, there were four 90-degree angles, yet all the figures were not similar. What is the role of angles in similarity?
- What does the word *corresponding* imply? Can one double the side length and double the angle measurement and still have two similar figures?
- How do the two components of similarity (corresponding angles and corresponding side lengths) provide a working definition of similarity?

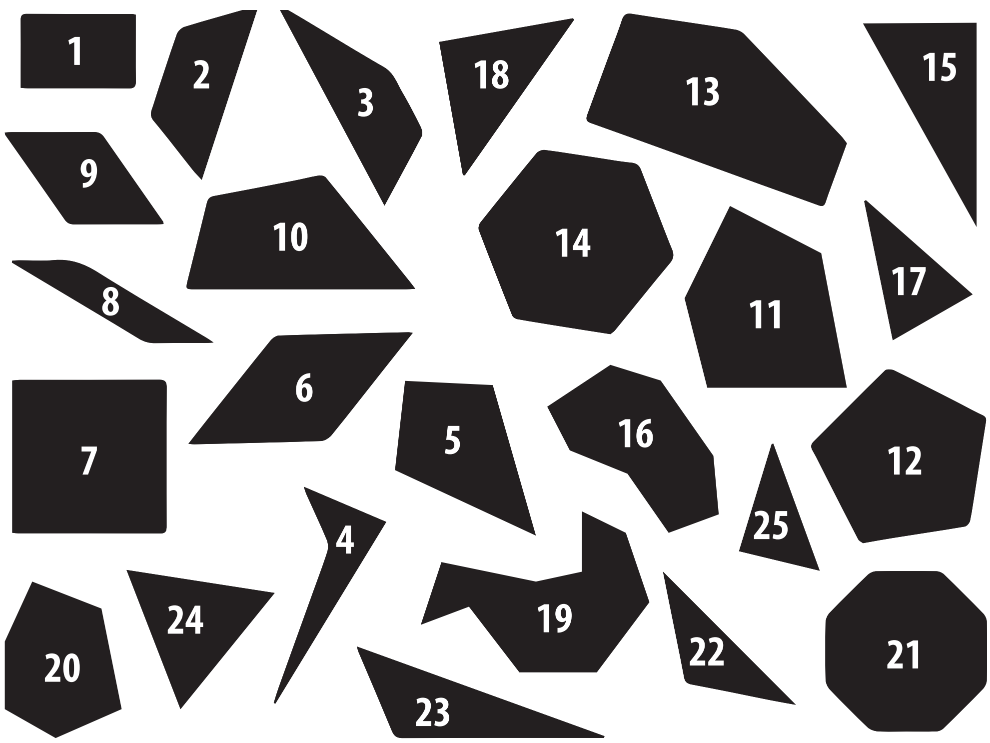
Have Student Teachers sketch three 90-degree angles on grid paper. These will become their reference angles. How might they:

- find 45-degree, 22.5-degree, and 135-degree angles by using one of the 90-degree angles that they drew?
- use another of their 90-degree angles to estimate angles of 60 degrees, 30 degrees, and 120 degrees?
- use their third 90-degree angle to find a 180-degree (straight) angle and a 360-degree (full rotation) angle?

Once Student Teachers have done this, ask them to describe the usefulness of a 90-degree angle as a reference, or benchmark, angle to help estimate and compare angles.

In the previous session, Student Teachers were told to bring their 'Attributes of Polygons' worksheet to class today. Have them look at the following polygons:

- Triangles: 18, 22, 24
- Quadrilaterals: 1, 2, 5, 6, 7, 9
- Pentagon: 12
- Hexagon: 14
- Octagon: 21



Assignment

Ask Student Teachers to use angles derived from the 90-degree reference angle to estimate the interior angles (and then the shape's angle sum) for each of the 12 polygons mentioned above. Have them bring the results to the next class session, where angles and angle sums will be explored more fully.

FACULTY NOTES

Unit 3/week 2: Angles, angles in polygons, 360-degree angles, tessellations

Session 1: Angles: Types of angles, measurement of angles, size of line segments not affecting angle size, interior angles of polygons, angle sums in triangles
Session 2: Angle sums in polygons, 360 degrees around a point
Session 3: Tessellations and tiling a plane

Faculty preparation for the upcoming week (1–2 hours)

- Read the following lesson plan:
  - 'What's Regular about Tessellations?':
    - ➢ http://tinyurl.com/Regular-Tessell
- Look through the following webpages:
  - Naming and measuring angles (video):
    - ➢ http://tinyurl.com/Benchmark-Angles-VIdeo
  - Angles around a point:
    - ➢ http://tinyurl.com/Angles-Around-Point
  - Why some polygons tessellate:
    - ➢ http://tinyurl.com/Tessel-Visual
  - Virtual pattern blocks (interactive applet; needs Java installed):
    - ➢ http://tinyurl.com/Virtual-Pattern-Blocks
  - Consistency of angle sum in a polygon with *n* sides (interactive applet):
    - ➢ http://tinyurl.com/Angle-Sum-Applet
  - Angle sum formula:
    - Dissecting polygons into triangles:
      - ➢ http://tinyurl.com/AlgLab-Angle-Sum
    - Using patterns to discover the angle sum formula (video):
      - ➢ http://tinyurl.com/Angle-Sum-Patterns
    - Developing the angle sum formula:
      - ➢ http://tinyurl.com/Angle-Sum-Formula1
    - Using the angle sum formula:
      - ➢ http://tinyurl.com/Angle-Sum-Formula2
      - ➢ http://tinyurl.com/Angle-Sum-Formula3
  - Naming tessellations:
    - ➢ http://tinyurl.com/Tessel-Naming

    (Interactive applet):
    - ➢ http://tinyurl.com/Tessellation-Applet
  - Rotating and flipping polygons to create tessellations:
    - ➢ http://tinyurl.com/Tessel-Tranformations
  - Patterns in Islamic art:
    - ➢ http://tinyurl.com/Tessel-Islamic-Art

- Distorting triangles, squares, and hexagons to design tessellations (four pages and interactive applet):
  - http://tinyurl.com/Tessellation-Applet-2
- Ripping triangle corners:
  - http://tinyurl.com/Ripping-Corners
- Naming and measuring angles (video):
  - http://tinyurl.com/Angle-Name-Measure
- Angle descriptions with photographs:
  - http://tinyurl.com/Angle-Types
- Angle descriptions with animation:
  - http://tinyurl.com/Angle-Animation
- Angle tutorial:
  - http://tinyurl.com/Angle-Tutorial
- There are 360 degrees around a point:
  - http://tinyurl.com/Angles-Around-Point1
  - http://tinyurl.com/Angles-Around-Point2

- Download and print out for your own use:
  - 'What's Regular about This Polygon?' (print out page one as a transparency for use in class):
    - http://tinyurl.com/Reg-Polygon-Transp
  - Angles around a point:
    - http://tinyurl.com/Angles-Around-Point
  - Coloured triangle-square tessellations (make three or four colour copies of these two pages to use as samples in small groups):
    - http://budurl.com/ColorTessell
  - Answer key to 'What's Regular about Tessellations' assignment:
    - http://tinyurl.com/Tessell-Answers
- Download and print out for class use and assignments (one copy per Student Teacher):
  - Pattern blocks:
    - http://tinyurl.com/Pattern-Blocks-PDF
  - Angles around a point:
    - http://tinyurl.com/Angles-Around-Point
  - Two semi-regular tessellation colouring sheets:
    - http://tinyurl.com/Tessel-Coloring-Sheets
  - 'What's Regular about Tessellations?' assignment:
    - http://tinyurl.com/Tessell-Handout
  - 'What's Regular about Tessellations?' cut-outs:
    - http://tinyurl.com/Tessell-Cut-Out
- Bring to class:
  - Analogue clock
  - Scissors
  - Paste or glue sticks
  - Chart paper
- Read through the plans for this week's three sessions.

Weeklong overview

This second week builds on Student Teachers' previous work with polygons in order to make explicit the types and characteristics of angles.

During Session 1, Student Teachers will learn to recognize and categorize angles based on their type as well as on their measurement in degrees. They will also consider why the length of the rays or line segments creating a given angle is not relevant to the size of the angle itself (which is the measurement in degrees of the interior space between the two rays or line segments that form the angle).

During Session 2, Student Teachers, building upon their work with angles to inform their understanding of polygons, will explore the interior angle sum of regular and irregular triangles, which eventually will lead to their finding the interior angle-sum of any polygon.

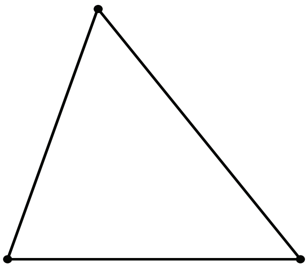
In Session 3, Student Teachers will engage in activities that will help them discover why certain polygons (or combinations of polygons) can tessellate (or tile a plane surface) without having any gaps or overlaps.

Unit 3/week 2, session 1: Angles

What do Student Teachers need to know?

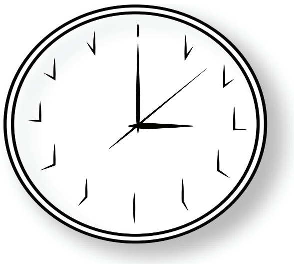
Types of angles. Angles are usually categorized by their measurement (acute, right, obtuse, straight, reflexive, whole). However, it is illuminating for students to shift their thinking and consider angles from a different perspective: wedge, branch, and dynamic.

A wedge angle is one whose two sides are closed by either additional sides (as in the first two pictures) or an arc (as in the bread). Once an additional side is added, the angle has become static.



A second type of angle can best be understood by looking at a tree branch. The angle formed has no additional side. It might be thought of as being composed of two rays that could go on forever if tree branches could grow to infinity.


Like the wedge angles, these open branching angles are static. However, there is a third type of angle that is dynamic: it moves around its vertex and is sometimes acute, sometimes straight, and sometimes obtuse. Think of a non-digital clock or watch. Note how the hands move around the centre, and as they do the angle between the hands changes minute by minute.



Take a look at this clock face, where the hours are not indicated in numerals, but in the angles the hour hand will make when the minute hand reaches 12. Notice the look of symmetry on the left- and right-hand sides of the clock face, and the gradual increase in the angle from 12 to 6 and its angles from 6 to 12 as reflex angles that go beyond the straight angle at 6 o'clock.

Angles are the space between two rays or line segments that meet at a vertex. A clock face can help students understand that the size of the angle formed by two rays or line segments converging at (or emanating from) a vertex is not dependent on the length of the line segments.

Consider two clocks: Big Ben and a small bedside alarm clock. The pictures below might imply the two clocks are the same size, but Big Ben (the largest four-faced chiming clock in the world) has a clock face of 7 metres (23 feet) in diameter. Compare that to the little alarm clock that probably has a diameter of 6 centimetres.

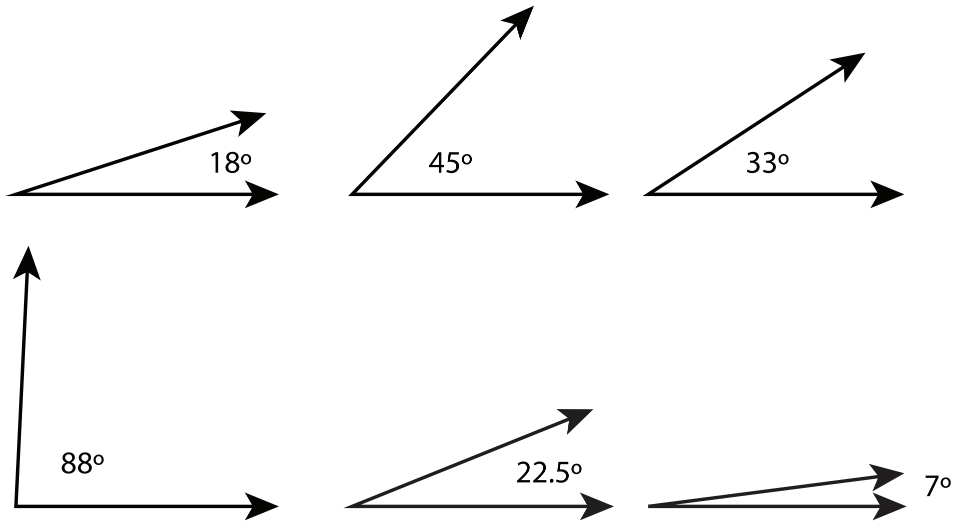
Yet both display the 30-degree angle indicating one o'clock. When considering angles, the size of the clock face and the length of the hands are irrelevant. The 30-degree angle remains constant.

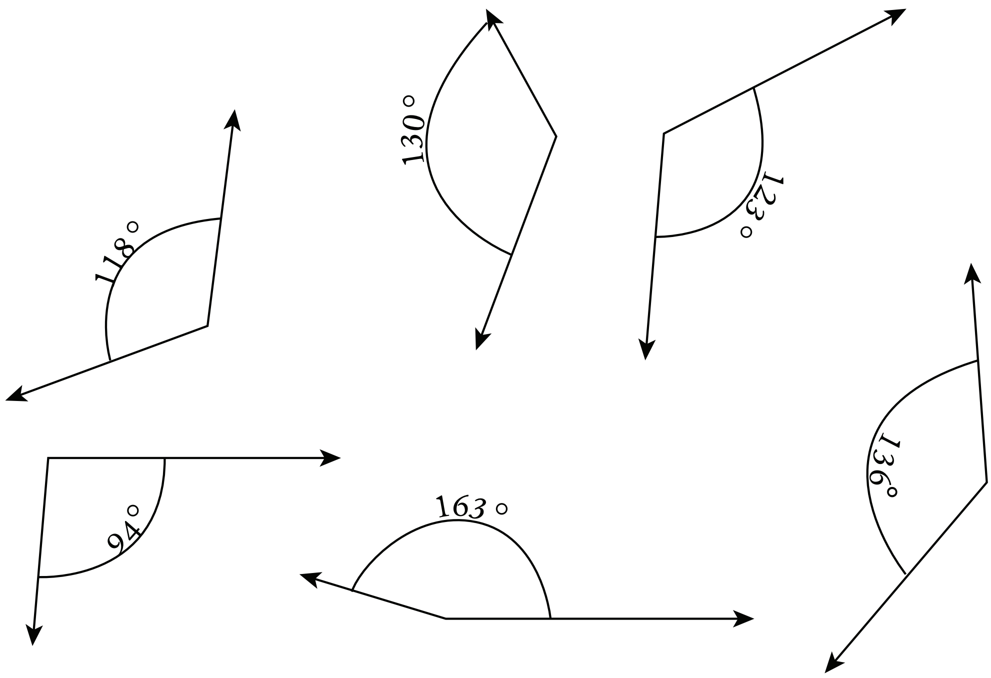
Angles can be categorized by measurement. Angles can be described by their relationship to two important benchmark angles: the 90-degree right angle and the 180-degree straight angle.

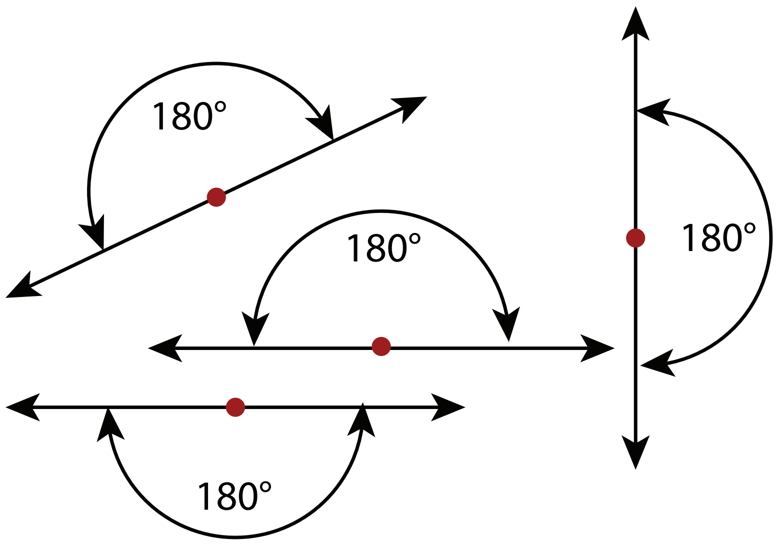
- Angles that measure more than 0 degrees and less than 90 degrees are termed *acute*. Thus, the benchmark angles of 30 degrees, 45 degrees, and 60 degrees are acute.
- An angle that measures exactly 90 degrees is called a *right* angle.
- An angle that is more than 90 degrees but less than 180 degrees is called *obtuse*.
- An angle that is exactly 180 degrees is called a *straight* angle.
- Angles greater than 180 degrees but less than 360 degrees are called *reflex* angles.

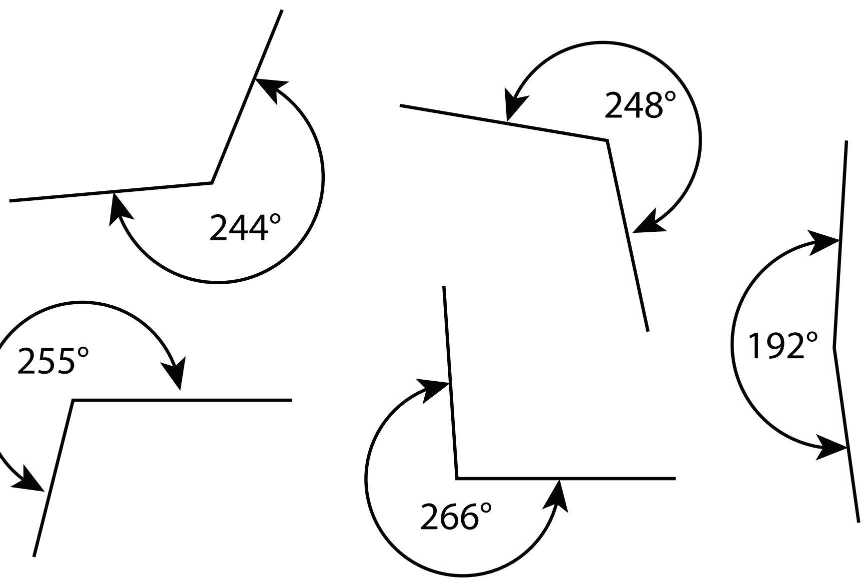
In one hour, the minute hand on an analogue clock moves through 360 degrees to create what is termed a *whole angle*, *round angle*, or *complete revolution*. The more formal term, *perigon*, is not necessary for students to know, but the idea of the 360-degree angle sum around a point will become important when considering circles and tessellations.

Here are images that display acute, obtuse, straight, and reflex angles. Note the angle measurement indicated for each.




118°
130°
123°
136°
94°
163°


180°
180°
180°
180°


244°
248°
255°
266°
192°

Interior angle sums of triangles. After Student Teachers have explored polygons and angles, they need to consider how these two geometric concepts are related.

If Student Teachers consider the interior angles of a regular polygon (such as an equilateral triangle or a square), what might they think?

Because they have been introduced to benchmark angles, they will probably conclude that an equilateral triangle has an angle sum of 180 degrees, as each of its three angles measures 60 degrees. They also will assume that a square and a rectangle have interior angle sums of 360 degrees because each figure has four 90-degree right angles.

What about a triangle with a 90-degree angle (a right angle triangle) or a scalene triangle with three different angles? What is the interior angle sum of those triangles? Is it also 180 degrees? Is the interior angle sum of 180 degrees true for all triangles?

Student Teachers may understand that a square and a rectangle are both quadrilaterals with an angle sum of 360 degrees. Is an angle sum of 360 degrees true for all quadrilaterals, such as rhombuses and trapezoids?

What about polygons with more than four sides? What is the angle sum of a regular hexagon?

How can Student Teachers use what they know about triangles to determine both the total number of interior degrees in a regular hexagon as well as the number of degrees in each interior angle?

Will this angle sum hold true for irregular hexagons as well?

As you pose these questions to the Student Teachers, realize that they do not need to know the answers now. There will be a homework assignment as well as a follow-up activity during the next class session to help them make generalizations and formalize their ideas.

How do children think about these concepts?

Although children may see angles in their everyday experience, they may need to be alerted to what they informally know.

For example, young children may walk down a street and see right angles in the street signs or in buildings. What about right angles in fabric? Why are right angles so prevalent in the man-made world around us?

What about the branching of trees as shown earlier? Or the movement of a clock's hands around a dial?

How can teachers help children notice and name the angles that are in their everyday environment?

Angles are the space between two rays or line segments that meet at a vertex.

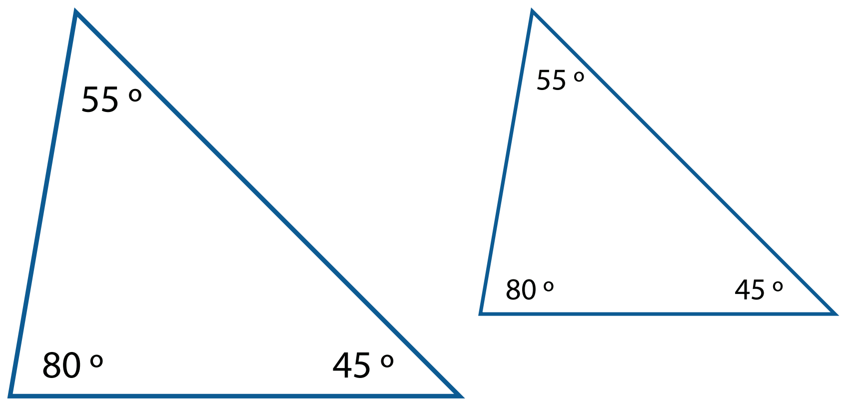
Children often incorrectly assume that the longer the line segments, the bigger the angle. This is because up to now children's experience with the words *bigger* and *larger* related to linear (one-dimensional) or area (two-dimensional) measurement.

If the following two triangles were drawn on a grid, the area covered by each angle in the first triangle would be bigger than the area covered by the second triangle's corresponding angles.

When applying their prior knowledge about size to angles, it is logical for children to assume that the angles in the first triangle below are bigger than the angles in the second.

It requires a major shift in their thinking for children to begin to consider a new kind of measurement (degrees) and a new definition of *bigger*.

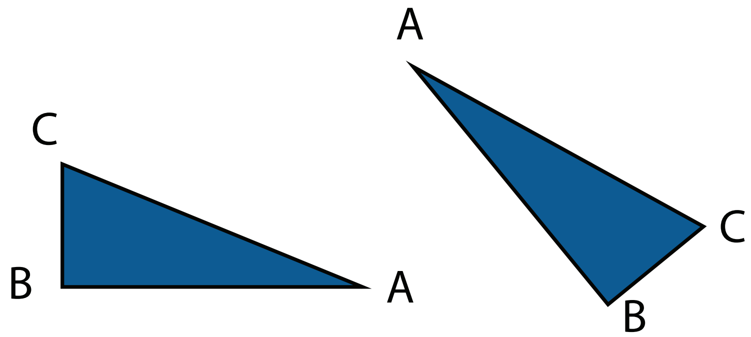
To help children understand this, they can cut out two similar triangles and lay the corresponding angles atop each other. They will see that the angles match up perfectly to each other and are therefore the same size—in degrees.



Angles can be categorized by measurement. When given pictures of acute, right, and obtuse angles, children intuitively know that they look different.

Children may describe a right angle as a square. (Note that the symbol for a right angle is a small square within the angle.) They may also say it looks 'like a corner' as many of the corners they see (such as pages of a book, tables, or windows) are right angles.

Just as young children needed to see that any three-sided polygon regardless of its orientation is a triangle, they now need to see that a right angle remains a right angle, even if its orientation is changed.



When describing acute angles, children often call them *pointy*, especially if they are less than 30 degrees.

Children usually lack ways to describe an obtuse angle, perhaps because these angles are less obvious in their everyday environment. Teachers need to identify real-life examples of obtuse angles, such as the hands of a clock at 10:10 or a stop sign.

Having children discuss their ideas about angles in simple shapes such as pattern blocks can also help them differentiate among acute, right, and obtuse angles.

Older children will be introduced to the idea of the 180-degree straight angle. This can be difficult if their informal definition of an angle means that an angle 'has a corner'.

This is where the activity of ripping the corners off a triangle and rearranging them to form a straight angle can illustrate that while a straight angle does not have a corner, it indeed has a vertex, which is where the three angles ripped from the triangle come together.

A discussion of straight angles by older children is an appropriate lead-in to determining the angle sum of a triangle.

Children may be comfortable knowing that triangles, even those that look different from each other, all have three sides and three angles.

But there is a third attribute of all triangles: the sum of the angles in any triangle equals 180 degrees. Telling children this fact is not enough. They need to experiment with a variety of triangles to prove that the three angles of any triangle will form a straight angle of 180 degrees.

This fact will become important as older children realize that they can subdivide any polygon into triangles and calculate the polygon's angle sum.

What is essential to do with Student Teachers?

- Introduce the three models for angles: wedge, branch, and dynamic.
- Assess if Student Teachers understand that the size of an angle is determined by its measurement in degrees, not the length of its two line segments.
- Angles can be categorized by measurement: although Student Teachers may be able to describe acute, right, and obtuse angles, they may not be familiar with the idea of straight, reflex, and whole angles.
- Have Student Teachers conjecture about the sum of the interior angles of a triangle, then have them prove or disprove their initial ideas in order to reach the generalization that all triangles have an angle sum of 180 degrees, not just special cases such as equilateral (60°, 60°, 60°) or isosceles right (90°, 45°, 45°) triangles.

Activities with Student Teachers

Types of angles: have Student Teachers recall what they said when describing angles in the pre-assessment in the first session of this unit.

Most likely they will refer to types of angles characterized by measurement in degrees. By drawing pictures similar to the ones above, introduce the idea of wedge angles within a closed figure (such as the triangle and bread slices) and angles that radiate from a vertex, such as branches of a tree.

Use the analogue clock to ask about angles formed when the minute hand moves around the clock's centre. Ask how they would determine the time if the clock face did not have numbers.

Finally, ask Student Teachers to name a real-life example for each of these three types of angles.

Ensure that Student Teachers understand the vocabulary required for the rest of the lesson, including *line segments*, *rays*, and *vertex*.

Have Student Teachers draw two squares of different sizes, then ask which square has the *bigger* angles. Student Teachers may be puzzled by this question and may ask what you mean by *bigger*. Ask what they think and to predict what a child would think. After hearing their answers, ask why a child may think that the angles in the larger square are bigger than the ones in the smaller square.

Ask Student Teachers why the angles in both squares are the same even though the size of the squares is different.

Draw two similar triangles on the board and ask the same questions.

Finally, draw two branching angles whose angle measurements are the same but have different length rays emanating from the vertex. Ask if these angles are the same. As the discussion develops, elicit from Student Teachers that angles are the space between two rays or line segments but that the length of the line segments or rays do not determine the size of the angle.

After Student Teachers understand that the size of an angle is not determined by the length of the rays or line segments, ask how they would determine if one angle is bigger than another.

Draw two acute angles of different sizes and ask which is the bigger angle, and why.

Draw a right angle and an obtuse angle and ask the same question.

Notice how Student Teachers describe their perceptions.

Ask: 'If the angles look different, how can we quantify their size and what sort of measurement system should be used?' If Student Teachers mention degrees, remind them of the benchmark angles they worked with in the previous session.

Many will know the terms *acute*, *right*, and *obtuse* and the angle measurements of less than 90 degrees, exactly 90 degrees, and between 90 degrees and 180 degrees.

When describing an obtuse angle, Student Teachers may say that it is 'greater than 90 degrees'. To challenge this, draw a straight line with a point (vertex) on it and ask them what sort of angle they see.

To demonstrate, draw two clock faces on the board and set one to show three o'clock. Ask which angle the hands make. Draw six o'clock on the second clock and ask which sized angle the hands make. Clarify that this is called a straight angle and its measure is 180 degrees.

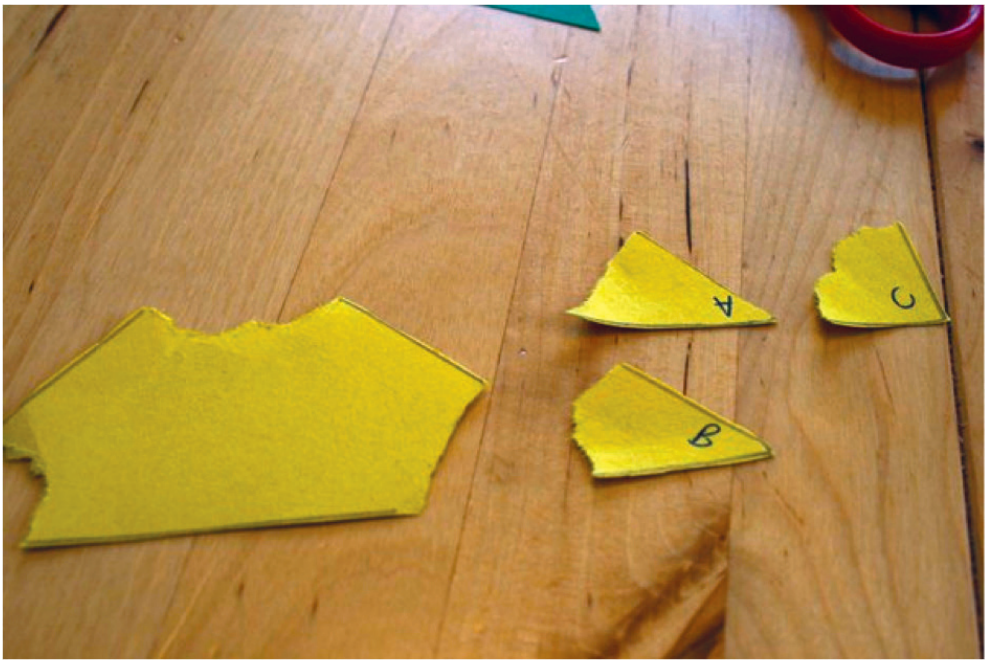
Ask how many degrees the hour hand will have gone through by nine o'clock. Note that angles greater than 180 degrees are called *reflex* angles, and when the hour hand has gone from midnight to noon, it will have made a complete rotation of 360 degrees.

Ask Student Teachers how many total degrees there are in the four angles of a square. What about in a rectangle?

Do they have any idea how many degrees there are in an equilateral triangle? In a right isosceles triangle?

At this point, distribute scissors and ask Student Teachers to cut out a triangle from notebook paper; label its vertices A, B, and C; and then rip off (not cut off) its vertices (refer to the webpage http://tinyurl.com/Ripping-Corners).

Their triangles will be of various shapes and sizes, which will be helpful for comparison purposes and support a generalization.



Ask them to rearrange the pieces. What do they notice? Is there more than one arrangement where the three angles are adjacent to each another and still form a 180-degree (straight) angle?

Ask if this is happening for all triangles regardless of size? Did anyone have a triangle for which the angle sum is not 180 degrees? Why or why not?

Finally, ask what generalizations they can make about the angle sum in a triangle. What do they believe to be the total number of degrees in any triangle? Why do they believe this is so?

Assignment

- Have Student Teachers colour the polygons on the pattern block template page (http://tinyurl.com/Pattern-Block-Template):
  - Hexagon (yellow)
  - Trapezoid (red)
  - Square (orange)
  - Triangle (green)
  - Thin rhombus (tan)
  - Other rhombus (blue)
- Then have them cut out the shapes and record two things:
  - The measure in degrees for each polygon's angles, without using a protractor.
  - Different ways that various shapes can be arranged around a single point to make a 360-degree angle. Have them use the 'angles around a point' handout (http://tinyurl.com/Angles-Around-Point) to record their discoveries.

(Student Teachers can also experiment with this activity by using the virtual pattern blocks from the website below.)

- Have Student Teachers cut out a variety of quadrilaterals (squares, rectangles, rhombuses, trapezoids, parallelograms, and non-defined four-sided figures).

  Using the same process that was performed with triangles, have them label and rip off the corners and arrange them. What patterns do they notice?

- Have Student Teachers cut out more quadrilaterals. Cut them in half on a diagonal. What do they notice about the resulting shapes? What about their angle sum? How does that angle sum relate to the angle sum they found in other quadrilaterals? Why do they think this is so?
- Have Student Teachers look through the following resources:
  - Angle descriptions with photographs:
    - http://tinyurl.com/Angle-Types
  - Angle descriptions with animation:
    - http://tinyurl.com/Angle-Animation
  - Angle tutorial:
    - http://tinyurl.com/Angle-Tutorial
  - Naming and measuring angles (video):
    - http://tinyurl.com/Angle-Name-Measure
  - Virtual pattern blocks (needs Java installed):
    - http://tinyurl.com/Virtual-Pat-Blocks
- Have Student Teachers bring the coloured-in cut-out pattern blocks to the next class session.

Unit 3/week 2, session 2: Angles in polygons, 360 degrees around a point

What do Student Teachers need to know?

Any polygon can be dissected into triangles to determine the sum of the polygon's interior angles.

This continues the discussion of all triangles having a 180-degree angle sum from the previous session.

To calculate the number of degrees in a given polygon, lines are drawn from one vertex to all the others. These are termed *diagonals*. The construction of these diagonals results in a series of triangles inside the polygon.

This process is called 'triangulating the polygon'. Because each of these triangles contains 180 degrees, the angle sum of the polygon can be calculated by multiplying 180 degrees by the number of triangles.

By calculating (and charting) the angle sum of three- through seven-sided polygons, a pattern will emerge. This pattern results in a formula that can be used to find the angle sum for a polygon of any number of sides.

This will result in a completed chart that looks something like this:

Polygon	Number of vertices (n)	Number of triangles	Angel sum (m)
Triangle	3	1	1(180) = 180
Quadrilateral	4	2	2(180) = 360
Pentagon	5	3	3(180) = 540
Hexagon	6	4	4(180) = 720
Heptagon	7	5	5(180) = 900
...	...	...	...
Decagon	10	8	8(180) = 1440
100-gon	100	?	?
***n*-gon**	n	$n - 2$	$(n - 2)180$

NOTE: It is crucial that the completed chart not be given to the Student Teachers beforehand. Because pattern detection is such an important mathematical trait, the goal is that the Student Teachers should not only learn the triangle sum theorem but should also develop this concept. Adequate time should be allotted for Student Teachers to develop this concept.

All polygons with the same number of sides have the same interior angle sum.

For homework, Student Teachers cut various quadrilaterals on the diagonal to form two triangles, each with an angle sum of 180 degrees. From this activity, Student Teachers should come to a generalization that this fact applies not only to squares (the regular quadrilateral) but to any quadrilateral.

When Student Teachers dissect polygons with various numbers of sides into triangles, they will come to the generalization that regardless of a polygon's shape or the measure of its individual angles, any polygon of n sides will have the same angle sum because it contains the same number of triangles.

For an interactive applet that demonstrates this angle sum consistency, see:

- http://tinyurl.com/Angle-Sum-Applet

If several polygons can be arranged around a point without any gaps, the sum of the angles surrounding the point is 360 degrees.

This continues the discussion of angles from the previous session in which acute, right, obtuse, and straight angles were emphasized. For homework, Student Teachers engaged in an activity using pattern blocks (whose angles measured 45 degrees, 60 degrees, 90 degrees, 120 degrees, and 135 degrees) to surround a point. Generalizations from this activity will help Student Teachers understand the concept of a whole angle. This concept is important as Student Teachers begin to work with tessellations in the next session.

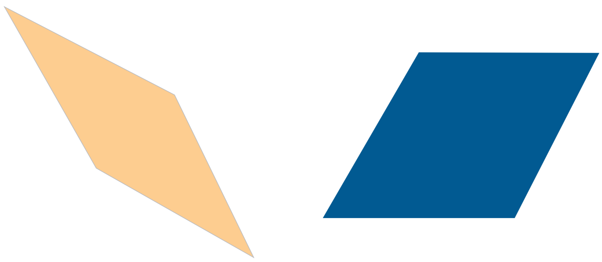
How do children think about these concepts?

When dissecting a polygon, children may not realize that the results need to be triangles. They may assume that any 'cut' will be valid.

Although most children would understand the word *diagonal* as related to the single line drawn between opposite vertices of a quadrilateral, they may not be aware of a more complete definition of *diagonal* as it applies to polygons with more than four sides.

This is an opportunity for teachers to clarify the more extended meaning of the word to mean two non-consecutive vertices of a polygon or polyhedron.

Children may assume that a 'skinny' rhombus (such as the tan pattern block on the left) has a smaller angle sum than the blue rhombus pattern block on the right. This is because their eyes are more apt to notice acute angles than obtuse ones.



This is another instance where, in order to build toward a generalization about the angle sum of all quadrilaterals, it is useful to have children actually rip off and arrange all four angles of a paper quadrilateral to show how its angle sum is still 360 degrees.

If children are asked to memorize the angle sum formula out of context, there is no guarantee they will remember it.

Thus it is important that they, like the Student Teachers in this course, actually chart data, notice patterns, come to a generalization, and only then develop a formula.

Not only will this process help them more fully understand the theorem, it will also allow them to reconstruct and use the formula when needed in the future.

There is an angle of 360 degrees around any point. Children may limit their thinking about angles to acute, right, and obtuse. Perhaps older children may consider straight and reflex angles.

But if they are given a point and asked to describe its surrounding angle, even adults are likely to say 0 degrees. (Just as they may have said '0 degrees' when introduced to a straight angle.) To their eyes, the angle around a point is invisible—unless given activities (such as those arranging pattern blocks around a point) to help them 'see' 360 degrees.

What is essential to do with Student Teachers?

- Dissect polygons into triangles to develop (not just provide) the angle sum formula.
- When given a variety of polygons with *n* sides, Student Teachers can demonstrate why all the polygons have a consistent angle sum.
- Show that there are always 360 degrees around a point.

Activities with Student Teachers

Begin by reviewing the homework.

Discuss the quadrilateral activity first. Ask about the Student Teachers' discoveries. Have Student Teachers extend their thinking by asking what would have happened if a square were not cut in half diagonally but crosswise into two rectangular halves. What was the angle sum of the original square? What is the angle sum of the two resulting rectangles?

Why did the two triangles, which resulted from cutting on the diagonal, have an angle sum equal to that of the original square, whereas cutting it crosswise doubled the angle sum?

Ask Student Teachers about their assignment to find the angles and angle sums for each of the pattern blocks. What was the angle sum of the square, trapezoid, and the two rhombuses? Why do they think that was so? What was alike about those four shapes?

Ask Student Teachers about the regular hexagon. What was the angle measurement at each of its vertices? How did they determine that? Did they use benchmark angles? Lay the corners of two equilateral triangles on the hexagon's internal angle. How did they find the hexagon's angle sum? Do they think this angle sum would be true for all hexagons (just as 180 degrees was true for all triangles and 360 degrees was true for all quadrilaterals)?

In this next activity, Student Teachers will develop the angle sum theorem. Again, it is crucial that they experiment and not be given the formula beforehand.

It may be necessary to define the term *diagonal* and to review that the angle sum of any triangle is 180 degrees. Divide the class into three groups. Have one group work with triangles and hexagons, the second group work with pentagons and decagons, and the third group work with quadrilaterals and octagons. (Note that each group has a second polygon with the number of sides double that of the first. Later you will ask if the angle sum doubles when you double the number of sides. If not, why not?)

Have the groups draw various polygons with a given number of sides, divide them into triangles with diagonals, and calculate the angle sum.

Polygon	Number of vertices (n)	Number of triangles	Angel sum (m)
Triangle	3	1	1(180) = 180
Quadrilateral	4	2	2(180) = 360
Pentagon	5	3	3(180) = 540
Hexagon	6	4	4(180) = 720
Heptagon	7	5	5(180) = 900
…	…	…	…
Decagon	10	8	8(180) = 1140
100-gon	100	?	?
n-gon	**n**	**$n - 2$**	**$(n - 2)180$**

Have Student Teachers report to create a chart similar to that above. Ask how doubling the number of sides related to the angle sum. Ask what patterns they notice. As they express their thoughts, ask what the pattern implies about finding the angle sum for a polygon with any number of sides.

Once they have discussed this in informal terms (but before they share a formula), ask Student Teachers to write a formula they could use to find the angle sum of any polygon. There are several ways to express this, such as $180(n - 2)$ or $(n * 180) - 360$. If different formulae emerge, use these formulae as an opportunity to show how they are equivalent expressions.

Return to the homework assignment about arranging pattern blocks around a point to form a 360-degree angle. Did some Student Teachers use only the paper cut-outs? Did others use the virtual pattern blocks from the Internet? What were their discoveries? How did using a recording sheet help organize their thinking and encourage predictions? What were their findings? Were there any surprises?

Assignment and resources

- Have Student Teachers bring crayons or felt-tip markers to the next class.
- Have Student Teachers explore the following websites that will review concepts taught in this lesson:
  - Angle sums of polygons by dissecting polygons into triangles:
    - http://tinyurl.com/AlgLab-Angle-Sum
  - Using patterns to discover the angle sum formula (video):
    - http://tinyurl.com/Angle-Sum-Patterns
  - Developing the angle sum formula:
    - http://tinyurl.com/Angle-Sum-Formula1
  - Using the angle sum formula:
    - http://tinyurl.com/Angle-Sum-Formula2
    - http://tinyurl.com/Angle-Sum-Formula3

- The angle sum is consistent for any polygon of *n* sides:
  - http://tinyurl.com/Angles-Around-Point
- There are 360 degrees around a point:
  - http://tinyurl.com/Angles-Around-Point1
  - http://tinyurl.com/Angles-Around-Point2

Unit 3/week 2, session 3: Tessellations, summary

What do Student Teachers need to know?

Some polygons, combination of polygons, or other non-polygonal shapes that are based on polygons can tile a plane and cover the surface without any gaps or overlaps.

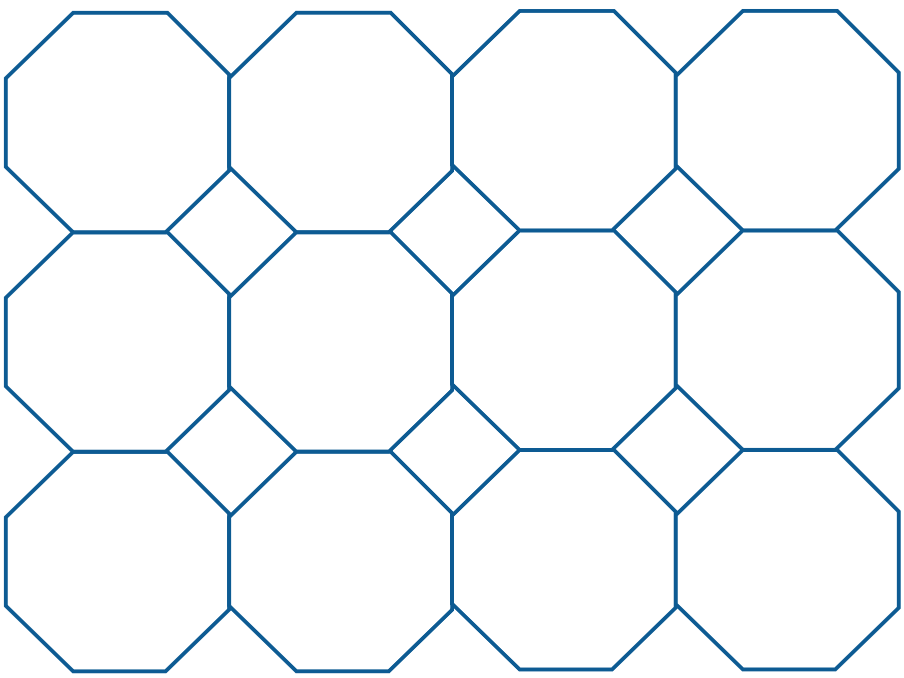
Those shapes are said to *tessellate* if they can be arranged in this manner.

Certain regular (equilateral, equiangular) polygons can tessellate using only themselves. These are called *regular tessellations*. There are only three of these regular tessellations: those composed of equilateral triangles, those composed of squares, and those composed of regular hexagons.

Here is a visual explanation of how these regular polygons form a regular tessellation:

- http://tinyurl.com/Tessel-Visual

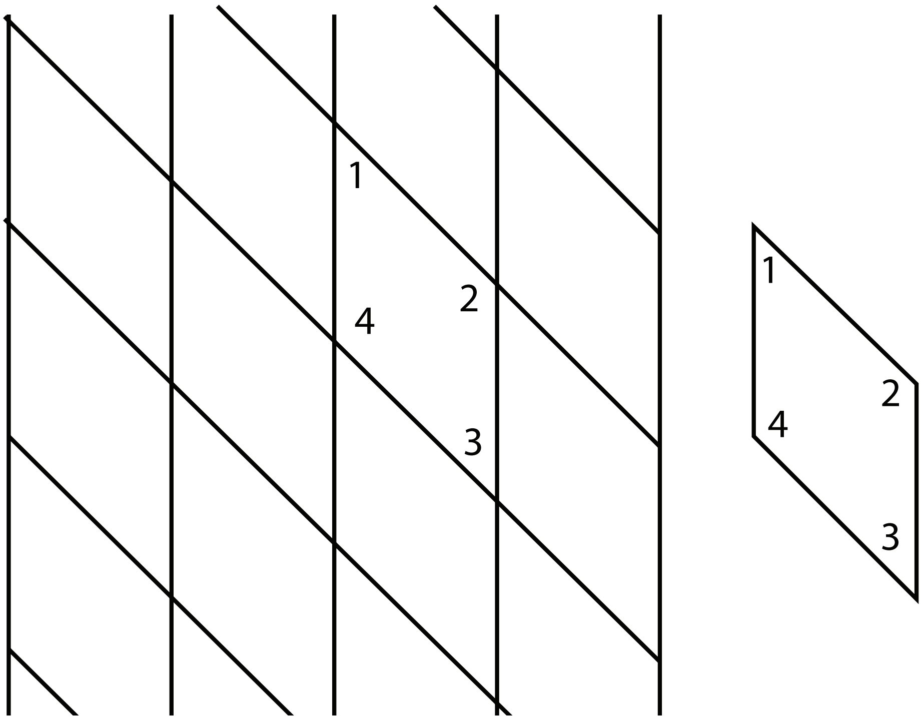
Semi-regular tessellations can be created by combining regular polygons, as in the octagon-and-square square design below.



Consider the case of a parallelogram, where sliding the shape is enough to create a tessellation. However, while all triangles and quadrilaterals can tile a plane, some may need to be rotated or flipped to form a tessellation.

See a visual about this at:

- http://tinyurl.com/Tessel-Tranformations



Still other tessellations can be created by distorting triangles, quadrilaterals, and certain hexagons to create carefully designed shapes, such as the polygons in the fish pattern and the non-polygonal shapes in M. C. Escher's men on horseback:

Tessellations depend on the previously discussed 360-degree angle around a point in a plane. Unless a point in a plane can be completely surrounded by shapes (without gaps or overlaps), a tessellation cannot exist. This is not to say that all arrangements of polygons around a point where the angles total 360 degrees are tessellations.

This is an important mathematical point for Student Teachers to grasp: the difference between necessary and sufficient requirements. While it is necessary that all tessellations require 360 degrees surrounding a point, not all combinations totalling 360 degrees around a point are sufficient to create a tessellation.

Tessellations are not simply an abstract mathematical construct. They are found in everyday life, such as in these floor tiles from ancient Pompeii (tessellating parallelograms) and this tile design from a modern tile catalogue (the octagon and square pattern).

Tessellations are especially prominent in Islamic art. Look at this slide show, 'Patterns in Islamic Art':

- http://tinyurl.com/Tessel-Islamic-Art

Tessellations beyond the regular and semi-regular are based on distorting certain basic shapes. For example, a square can squashed to become a rhombus. Then the opposite sides of that rhombus can be extended to different lengths to become a generalized parallelogram.

How do children think about these concepts?

When asked to use polygons to tile a plane—covering it with polygons so that there are no gaps between shapes—children instinctively think about squares, such as those on a chessboard.

Because of their lack of experience, however, children do not realize that any triangle or quadrilateral can be repeated to tile a plane. If given a cut-out of a quadrilateral (such as a parallelogram), children can manipulate and trace it repeatedly to determine that the shape can tile a plane.

If children are given a familiar shape (such as a regular octagon), can they predict its ability to tessellate? We often ask children to make similar predictions (in many areas of mathematics) without hands-on experience. In fact, if shown a regular octagon (a stop sign shape), most adults assume it will tessellate. It is only by manipulating and tracing a cut-out of a regular octagon that they discover that it does not form a regular tessellation.

Just as children need to be alerted to the polygons and angles they see every day, they also need guidance in seeing tessellations in their everyday world.

Regular tessellations of squares are as common as floor tiles, or perhaps a chess-board. Children may be familiar with the arrangement of hexagons and pentagons on a football (although the design is on a sphere rather than a plane). Teachers should continue to find opportunities to point out tessellations both in art and in their students' environment.

Children can design their own tessellations even before they understand the 360-degree rule. Even children as young as seven can be given instructions that will allow them to create a tessellation from one of the basic tessellating shapes.

What is essential to do with Student Teachers?

- Some shapes can tessellate to tile a plane, whereas some cannot.
- Three regular polygons can tile a plane, resulting in regular tessellations. Several other regular polygons can be combined to create semi-regular tessellations.
- Tessellations are based on shapes creating a precise 360-degree angle around a point, with no gaps or overlaps.
- Tessellations are common both in real life (such as floor tiles) and in art.
- New tessellations can be formed by distorting a basic geometric shape: triangle, quadrilateral, or hexagon.

Activities with Student Teachers

Begin class by dividing Student Teachers into groups of four so they can use the cut-out pattern blocks they prepared.

To introduce the concept of tessellations, refer to the 'angles around a point' activity that Student Teachers discussed in depth during the previous session. Have Student Teachers recall which one-colour pattern blocks could be used to surround a point. Ask which of those were regular (equilateral, equiangular) shapes.

Do Student Teachers realize the distinction between the three regular polygons—the (green) equilateral triangle, (orange) square, and (yellow) regular hexagon—versus the (blue) rhombus and (red) trapezoid?

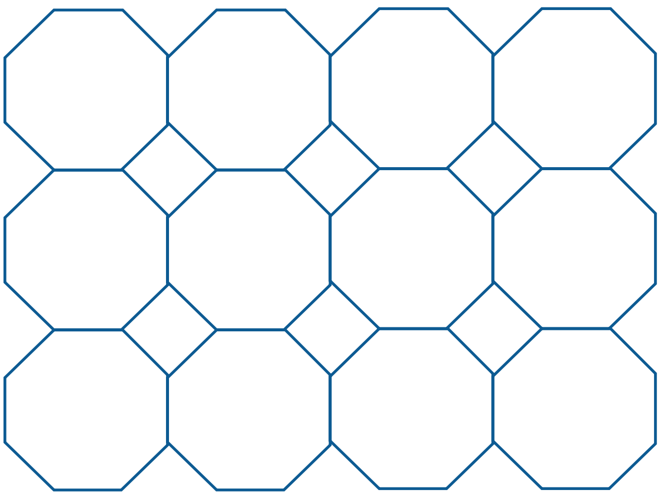
> NOTE: Even if the blue rhombus and red trapezoid are not regular polygons, they are still quadrilaterals and as such, with reorientation, can tile a plane.

To extend their around-a-point experience, introduce the terms *tessellation* and *tiling the plane*, defining them as necessary to make sure Student Teachers understand that tessellating shapes cannot overlap nor have gaps between them.

Distribute the pattern block cut-outs and have Student Teachers work in their groups to find the regular shapes that tessellate. They should find that the triangles, squares, and hexagons can tessellate. Describe these as regular tessellations.

Next, ask them if the triangle and square be used together to create a tessellation. If a group discovers only one way that the shapes can form a tessellation, challenge them to find another. (These will be shown in the handouts for the next activity.)

Explain that a tessellation that occurs when more than one regular polygon can create a tessellation is called a *semi-regular* tessellation. Even though there is no pattern block for a regular octagon, have Student Teachers sketch a lattice of regular octagons that are connected to each other. What do they notice about the spaces between the octagons? What type of tessellation is this?



Give Student Teachers the opportunity to work with the remaining pattern blocks to discover if they can create tessellations with them. Because the two rhombuses and the trapezoid are quadrilaterals, they can tessellate. However, because they are not regular polygons (equilateral and equiangular), the results are neither regular nor semi-regular. These types of tessellations are termed *non-regular* tessellations.

To follow up on the activity in which Student Teachers used equilateral triangles and squares to create semi-regular tessellations, distribute the following two worksheets where two different designs result from triangles and squares:

- http://tinyurl.com/Tessel-Coloring-Sheets

Have Student Teachers circle one 360-degree angle on each sheet, then note the polygons that create the angle. How many sides does each have? Have Student Teachers write that number in the shape. What do they notice about the numbers? (There are four three-sided shapes and two four-sided shapes.)

Ask what is different about the numbers on the two sheets. (They are in a different order.) Have Student Teachers note the header on each page, explaining that they just discovered the notation by which a particular arrangement of equilateral triangles and squares can be specified.

To finish this activity, distribute the colour copies made from http://budurl.com/ColorTessell to each group. Have each Student Teacher in the group choose a different colour pattern and quickly colour in one of their black-and-white handouts. As they begin to complete this colouring activity, ask how the tessellation takes on a different look than when it was simply black lines on white paper.

Remind Student Teachers that when working with children, using colour to highlight is an important instructional practice.

Dedicate time at the end of this session to engage in a Student Teacher–led summary of what they learned about polygons and angles during the past two weeks. Compare their current responses to those that were recorded during the pre-assessment on the first day of Week 1.

What new things have they learned? Which of their personal misconceptions were addressed? How was the manner in which they learned about polygons and angles different from the way they were taught about these topics in secondary school?

Assignment and resources

When distributing the regular and semi-regular tessellations assignment 'What's Regular about Tessellations?' (http://tinyurl.com/Tessell-Handout and http://tinyurl.com/Tessell-Cut-Out), note that the questions on the homework sheet begin with angle sums for *n*-gons, with the additional stipulation that these are regular (equilateral, equiangular) polygons.

How can Student Teachers use what they know about the triangle dissection of polygons to find the number of degrees for each of the angles in a regular polygon?

After they complete the chart, have them use this interactive applet to answer the remaining questions:

➢ http://tinyurl.com/Angle-Sum-Applet

Have Student Teachers read through the following websites to review concepts discussed in class. Make sure they know to click on the images at the bottom of these pages to see real-life examples.

- Regular tessellations:
  ➢ http://tinyurl.com/Tessel-Regular
- Regular and semi-regular tessellations:
  ➢ http://tinyurl.com/Tessel-SemiRegular
- Tessellations of quadrilaterals:
  ➢ http://tinyurl.com/Tessel-Quadrilat

Have Student Teachers review additional coloured-in tessellations at:

➢ http://budurl.com/ColorTessell

Aas well as the slideshow on patterns (including tessellations) in Islamic art:

➢ http://tinyurl.com/Tessel-Islamic-Art

FACULTY NOTES

Unit 3/week 3: Geometric measurement – Area, perimeter, relationships between area and perimeter

Session 1: Introduction to area
Session 2: Introduction to perimeter
Session 3: Exploring the relationship of area to perimeter and vice versa

Faculty preparation for the upcoming week (1–2 hours)

- Review the following website:
  - Finding area and perimeter of irregular shapes:
    - http://tinyurl.com/Area-Perimeter-Irregular
- Download and print out for Student Teacher use:
  - Centimetre grid paper:
    - http://tinyurl.com/GridCm
- Bring to class:
  - Graph paper
  - Square tiles (or have Student Teachers cut squares from centimetre grid paper)
  - Rulers
  - Scissors
  - String
  - Masking tape
  - Assorted boxes to measure (Student Teachers should be asked to bring these to class)
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 this week begins with the concept of area. Many textbooks start a unit on geometric measurement with perimeter because it is one-dimensional linear measurement.

But children do not instinctively think about measurement around something. Rather, living in a world where they see surfaces, they think two dimensionally, intuitively understanding area. This is similar to how angles and polygons were approached earlier in this course—with the way children think rather than the traditional textbook format, which often begins with the more abstract concept of angle and then moves to polygons, which children see in real-life situations every day. Therefore, teachers are advised to begin with polygons so that children will begin to see how angles relate to the shapes they see around them.

This is why this third week of the 'Geometry' unit begins with area, then moves to perimeter, and finally explores the relationship between the two.

Two-dimensional shapes are not all polygons. They might be irregularly shaped, such as a lake, one's hand, or a circle. When trying to measure the area of these shapes, it is important that Student Teachers understand the major mathematical concept that a measurement is always an estimate.

Measurement also depends on determining a unit of measurement (which could be a standard unit such as a centimetre or non-standard unit such as a paper clip) and then repeating that unit. For adults, this seems obvious, but confusion about this occurs not only with small children but also with older children.

Session 2 is devoted to perimeter. The classroom approach begins with a hands-on experience: exploring perimeter by wrapping objects with string and then using a ruler to measure the string's length to determine perimeter.

Session 3 addresses the relationship between area and perimeter. The idea of 'same area, different perimeter' will be explored in a simple manner by using graph paper. If I trace my hand with fingers closed, I can estimate my hand's area and perimeter. But what happens when I extend my fingers? What does this new tracing look like? What is the area of this new drawing? What is its perimeter? How are the two tracings the same? How are they different?

To explore the opposite concept, 'same perimeter, different area', something as simple as a large loop of string or yarn can be illustrative. The string is a given perimeter. But when held by a group of four Student Teachers, it can be shaped into quadrilaterals with various areas.

Unit 3/week 3, session 1: Geometric measurement – Area of irregular shapes and polygons

What do Student Teachers need to know?

Area is a measurement of a two-dimensional surface and is expressed in square units.

All two-dimensional surfaces (irregular figures, polygons, circles, and ellipses) can be measured to estimate their area.

Various methods and tools can be used to estimate area.

All measurements (not just of area) are considered estimates and depend on the level of accuracy of the tools used.

Measurement depends on determining a unit of measure and then repeating that unit until an estimated measurement has been obtained.

Formulae for finding the area of a rectangle can be used to determine the area of other polygons such as triangles, parallelograms, and trapezoids.

How do children think about these concepts?

Children initially tend to confuse the concepts of area and perimeter. Once they acquire these misconceptions it is extremely difficult for children to unlearn them. Therefore they need extensive, hands-on activities in measuring area and perimeter so that they have a clear understanding of how area and perimeter are different and what each of these measurements entails.

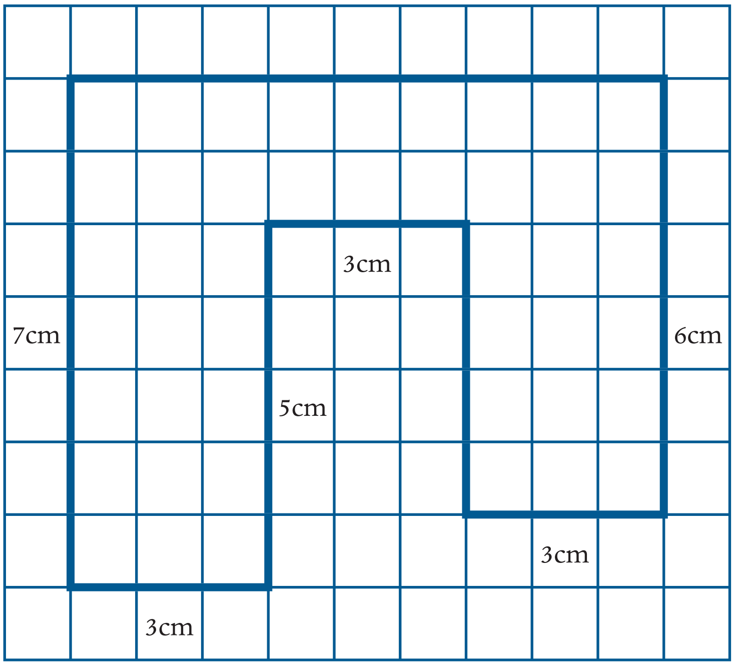
If area is introduced in textbooks beginning with squares and rectangles with their dimensions labelled, children tend to assume that area only applies to squares and rectangles.

This is why it is important to begin with the generalized concept of area as the measurement within any shape, and why this topic begins by measuring the area inside irregular shapes.

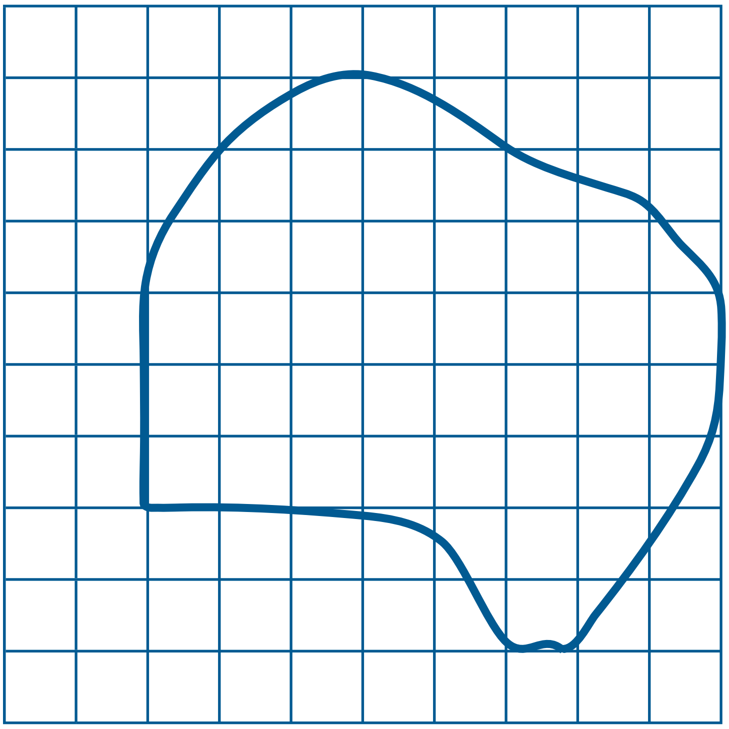
Once children hear that area is expressed in square units, they assume that only square (or at least rectangular) surfaces have area. This is another reason to begin with the area of irregular shapes. Beginning with the area of irregular shapes also leads to the idea that circles (which are certainly not square) have an area measured in square units.

Textbooks often label rectangles and squares with their dimensions, allowing children to simply multiply the given side lengths to find area. However, even if children are able to do this accurately, it does not guarantee that they understand the fundamental concept of area.

Primary school textbooks usually present formulae for area: length times width for rectangles, side times side for squares, and ½ (base times height) for triangles. This is often extended to polygons that are divided into simpler shapes, where each sub-shape's area can be calculated, with the resulting measurements added to find the polygon's total area.



However, this formulaic model does not allow for calculating/estimating the area of non-polygonal two-dimensional shapes, such as this lake:



What is essential to do with Student Teachers?

- Introduce area as a generalized concept, not simply a way of measuring the surface of polygons.
- Have Student Teachers use various tools, strategies, and formulae to determine the area of irregular shapes, various quadrilaterals, compound polygons, and triangles.
- Emphasize the three fundamental aspects of measurement:
  - Choosing a unit
  - Using the unit repeatedly when measuring
  - All measurements are estimates, depending on how fine-grained the measuring instrument is.

Activities with Student Teachers

Distribute grid paper and have Student Teachers draw an irregular shape on the grid. Ask them to work with a partner to estimate the shape's area. Notice how they address both the 'whole squares' and 'partial squares' inside the shape. Are they counting whole squares and combining partial squares to make a whole?

Bring the class together for a whole group discussion of their ideas. Begin by reminding Student Teachers that their measurements will be approximations. Then ask how their estimations could be made more accurate. If no one mentions it, ask about the size of the grid. How would their measurement differ if the same shape were drawn on traditional quad graph paper?

Some Student Teachers may try the following methods using a formula they learned in school rather than by counting:

- Drawing a rectangle closely around the shape, measuring the length and width of the rectangle, and then multiplying these two measurements. The area of the shape would be less than that of the rectangle.
- Drawing a rectangle close to the edge inside the shape, measuring the length and width of the rectangle, and then multiplying these two measurements. The area of the shape would be greater than that of the rectangle.

If no one suggests using a formula, do not introduce it now. Instead, ask Student Teachers to draw a rectangle or square on their grid paper, labelling the shape's sides with measurement in centimetres. Notice how they calculate the area now. Do they count? Or do they multiply the measurements of the adjacent sides? Remind them of the work they did with factors and multiples. How does this relate to finding the area of rectangles and squares? Can they come up with a formula for calculating the area of these shapes?

Next, ask Student Teachers to draw a right triangle on their grid paper. Have them work in pairs to find the area of the triangle without using a formula. Notice if anyone draws a similar right triangle on the hypotenuse of the original, forming a rectangle whose area they now can calculate.

When they are finished, ask the whole group to discuss their methods. Some Student Teachers may say they recalled the formula for the area of a triangle that they learned children. Probe further, asking questions such as:

- Why did you use the words *base* and *height*?
- Did you draw a second triangle on the hypotenuse? What shape resulted?
- How do terms *base* and *height* relate to the rectangle's length and width?
- Can this formula only be used for right triangles?
- If this formula can be used for other triangles (or all triangles), how can you prove it?

Finally, ask Student Teachers to draw a non-rectangular parallelogram on their grid paper. Mention that their parallelogram, like the triangle they drew, does not have length and width. Ask them to look carefully at the shape and find a way to make only one cut so that when the two pieces are rearranged, they can be taped together to create a rectangle. What do they notice? Can they create a generalized formula for finding the area of a parallelogram by using the words *base* and *height*?

Mention that their homework assignment will include reading about how to use the area of simpler shapes to calculate the area of compound shapes.

Assignment

Have Student Teachers visit this website that explores finding the area of irregular shapes and using the area of simple shapes such as squares and rectangles to determine the area of compound shapes:

- http://tinyurl.com/Area-Irregular

Have each Student Teacher bring an empty box to the next class session. These boxes will be used to explore the concept of perimeter as well as next week's topics: surface area and volume.

Unit 3/week 3, session 2: Geometric measurement, perimeter of polygons, and irregular shapes

What do Student Teachers need to know?

Perimeter is a one-dimensional (linear) measurement that surrounds a two-dimensional figure or three-dimensional object.

Perimeter can be measured around irregular figures, polygons, circles, and ellipses (and three-dimensional objects).

Various methods and tools can be used to find perimeter.

Formulae for finding perimeter may be expressed in various ways.

Although it may be relatively easy to estimate the area of certain basic shapes (such as triangles, trapezoids, and parallelograms other than squares and rectangles), it is significantly more difficult to estimate their perimeter. This is because these shapes have sides that do not meet at right angles. (Or as children would say, they have sides that are 'slanted'.)

If Student Teachers enquire about the perimeter (circumference) of circles, note that this concept will be explored in detail next week.

How do children think about these concepts?

If perimeter is introduced by measuring and adding the dimensions of squares and rectangles or by using a formula, children tend to assume that perimeter is a characteristic of polygons.

This is why it is important to begin with the generalized concept of perimeter as the measurement around any shape. Thus, this topic begins with measuring the perimeter of irregular shapes and then moves to discovering the area of various polygons. (Later in this unit Student Teachers will investigate the perimeter of circles, a measurement we call *circumference*.)

Textbooks often label shapes with their dimensions, making it relatively simple for children to add the side lengths to find the perimeter.

However, this is not how measuring perimeter occurs in the real world. In everyday situations involving perimeter, the dimensions are not known. They need to be measured by using various tools: rulers or measuring tape, string for irregular shapes (which then needs to be measured by a ruler or tape-measure), and for large shapes (such as the classroom floor) by a tool such as a trundle wheel.

Practicing these types of hands-on measurement activities in the classroom (and outside on the street or play area) not only prepares children to work with perimeter in real-life situations, but also prepares them for generalizing their real-life experiences into mathematical formulae.

Textbooks may give only limited formulae for perimeter. Children need to understand that the textbook's formula is only one of several valid methods of expressing a way to calculate perimeter from given dimensions.

For example, textbooks may express the formula for the area of a square as 4S, implying multiplication. However, the additive formula S + S + S + S is equivalent to 4S and is equally valid.

Similarly, the perimeter of a rectangle can be expressed as L + L + W + W as well as 2L + 2W or 2 (L + W).

As children begin to develop their own ways of expressing formulae for perimeter, teachers need to emphasize the notion of equivalent expressions, an important mathematical concept addressed in the 'Algebra' unit.

As mentioned above, children often have a real difficulty perceiving that the slanted side of a figure cannot be measured by counted squares. This is not so much of a mis*conception* as it is a mis*perception*. Later, this results in the older student's common misconception of the 'equilateral right triangle'. Try to envision how a child might come to this mistaken conclusion.

What is essential to do with Student Teachers?

- Introduce perimeter as a generalized concept, not just a way of measuring around polygons.
- Have Student Teachers use various tools to determine the perimeter of irregular shapes, regular polygons, and irregular polygons.
- Have Student Teachers develop multiple formulae to express equally valid ways of calculating perimeter.

Activities with Student Teachers

To help Student Teachers experience the fact that irregular shapes (not only polygons) have perimeter, draw a large irregular shape on the board. Ask the class for suggestions about how they might estimate the shape's perimeter. Be prepared for confusion. At this point display the string, masking tape, and the ruler or tape-measure. Ask Student Teachers how they could use these tools to determine the shape's perimeter. Some may suggest placing the string on the perimeter, securing it with tape where necessary, removing the string, and then measuring it with the ruler or tape-measure.

Ask questions such as:

- If you found the original question difficult to answer, why?
- How did simply seeing the string, tape, and measuring tool spur your thinking?
- Would it have been better to have shown or given you the tools before asking the original question? Why or why not?
- Do you think the measurement you found was a reasonable estimate of the shape's perimeter?
- How could you have increased the measurement's accuracy?
- What are some other ways children could measure the perimeter of an irregular shape? Suppose the shape were drawn on a horizontal surface such as the floor rather than an upright surface such as the board? (A sample suggestion might be to surround the shape with a non-standard measurement tool such as a paperclip, and then multiply the length of a single paperclip by the number of paperclips used.)

After the previous activity, using a string and a ruler (or simply a tape-measure) to measure the perimeter around a three-dimensional object such as the boxes Student Teachers brought to class may seem overly simple. However, this is an opportunity to stress that not only a flat, two-dimensional figure such as the one drawn on the board has a perimeter, but so do three-dimensional objects.

Student Teachers can also be asked to find the perimeter of the classroom. Ask which tool (paperclips, a 12-inch ruler, a yardstick or metre stick) they think would be most useful in determining the most accurate measurement for the perimeter of a sheet of paper, their desk, and the classroom. Because the paper, the desk, and the classroom are most likely rectangular, can they think of an efficient way to calculate perimeter? Student Teachers should come up with a several formulae such as (2 x length) + (2 x width) or 2 x (length + width).

Ask Student Teachers to draw a triangle, a parallelogram that is not a rectangle, and a trapezium on grid paper. Each of these shapes has at least two sides that do not meet at right angles. How might they determine the perimeter of these shapes? Notice if they refer to what they learned earlier in the session about the perimeter of irregular shapes or if they try to develop a formula to calculate the line segments that are not perpendicular.

Finally, ask Student Teachers:

- Which measurement did you think was easier to calculate, area or perimeter?
- Note that textbooks sometimes introduce perimeter before area. What do you think of this instructional sequence?
- Should perimeter and area be taught together or separately?
- How was the hands-on nature of these activities different from calculating the perimeter of shapes in textbooks that are already labelled with measurements?
- Why do you think children confuse area and perimeter?
- What might you do as a classroom teacher to ensure that children understand the difference between these two measurements?

Assignment

To be determined by the Instructor.

Unit 3/week 3, session 3: Relationship between area and perimeter

What do Student Teachers need to know?

Area and perimeter are two different types of measurement.

Shapes with a constant perimeter can vary in their area.

Shapes with a constant area can vary in their perimeter.

The mathematical concepts of maximum and minimum can be visualized when working with area and perimeter and charting the resulting measurements.

How do children think about these concepts?

Even when children understand the difference between area and perimeter, they may assume that any rectangle with an area of, for example, 24 square units will have a fixed perimeter.

This is why it is important to have them experiment with making all possible rectangles with a fixed area and whole number sides. This will allow them to see that differently shaped rectangles (with different perimeters) can be made from a given number of square units.

Research has shown that it is more difficult for children to hold perimeter constant and area variable. This is why it is important to have Student Teachers use a variety of techniques to explore their own understanding of rectangles with the same perimeter but different areas.

What is essential to do with Student Teachers?

- Student Teachers will explore the relationship of area and perimeter from two different perspectives:
  - Area as a constant with varying perimeters
  - Perimeter as constant with varying areas
- Have Student Teachers use various tools, strategies, and formulae to determine the area of irregular shapes, various quadrilaterals, compound polygons, and triangles.

Activities with Student Teachers

Have Student Teachers trace one of their hands on grid paper, first with the fingers closed, then with the fingers spread apart. (This will help them visualise a constant area with different perimeters.)

Use a given number of square tiles (constant area) and arrange them in different configurations to design polygons with different perimeters.

Use a two-metre loop of string or yarn (constant perimeter) and have four Student Teachers hold it to create rectangles of different areas.

Have Student Teachers chart the results of their findings to discover patterns that imply minimum and maximum.

Assignment

Have Student Teachers bring cylindrical objects and boxes to the next class session.

FACULTY NOTES

Unit 3/week 4: Geometric measurement – Circles (circumference and area), surface area of cuboids and cylinders

Session 1: Developing an understanding of pi (π) and its use in determining the perimeter (circumference) of circles
Session 2: Using the concept of radius squares to approximate the area of circles
Session 3: Introduction to the surface area of cubes, cuboids, and cylinders

Faculty preparation for the upcoming week (1–2 hours)

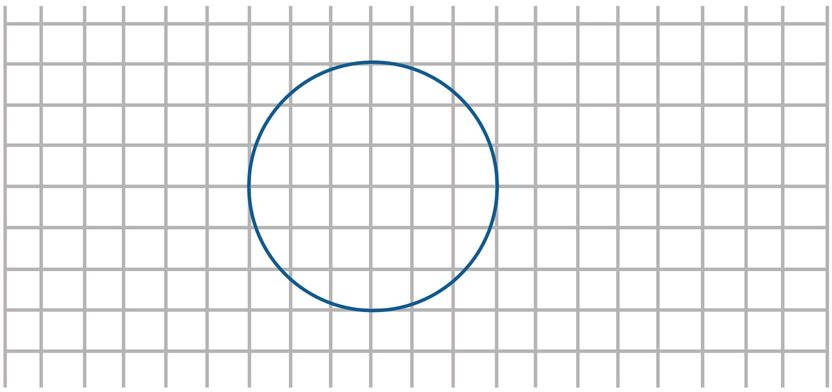
- Look through the following websites that address circumference, area of a circle, and surface area of cuboids and cylinders:
  - Finding circumference:
    - http://tinyurl.com/FInd-Circumference
  - What is a 'radius square'?
    - http://tinyurl.com/What-Is-Radius-Sq
  - The parallelogram method for approximating the area of a circle:
    - http://tinyurl.com/Circle-Parallelogram-1
  - Interactive website that addresses surface area and volume:
  - http://tinyurl.com/Surface-Area-vs-Volume
- Download and print out for Student Teacher use:
  - Centimetre grid paper:
    - http://tinyurl.com/GridCm
  - Radius squares:
    - http:/tinyurl.com/Radius-Square-Cut-Out
- Bring to class:
  - Tape-measures
  - String
  - Rulers
  - Scissors
  - Glue sticks
  - Compasses
  - Assorted cylinders and boxes (have Student Teachers bring these to class)
- Read through the plans for this week's three sessions.

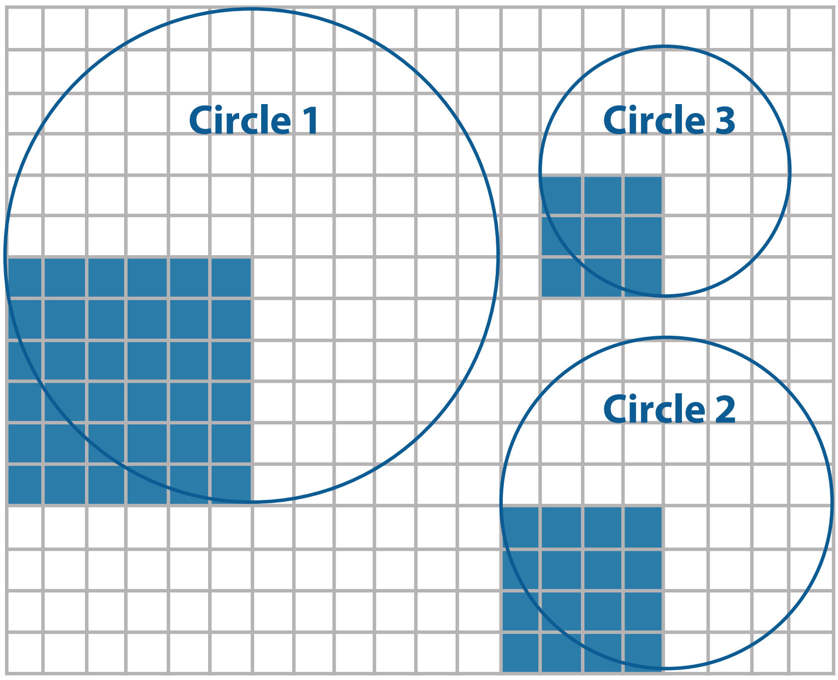
Weeklong overview

Session 1 begins by using string to surround various cylinders and comparing the length of the string to a cylinder's diameter. By charting the results, Student Teachers should come to the conclusion that there is a relationship between the diameter and the perimeter of the cylinder (which is called *circumference*). They may express this relationship as 'three and a little bit more', which will help them understand pi.

During Session 2, Student Teachers will extend what they learned about pi to find the approximate area of circles.

Just as they explored circumference prior to using a formula, Student Teachers will explore the area of circles by using a counting method and a novel visualization (radius squares) to make sense of the formula A = pi (r^2).

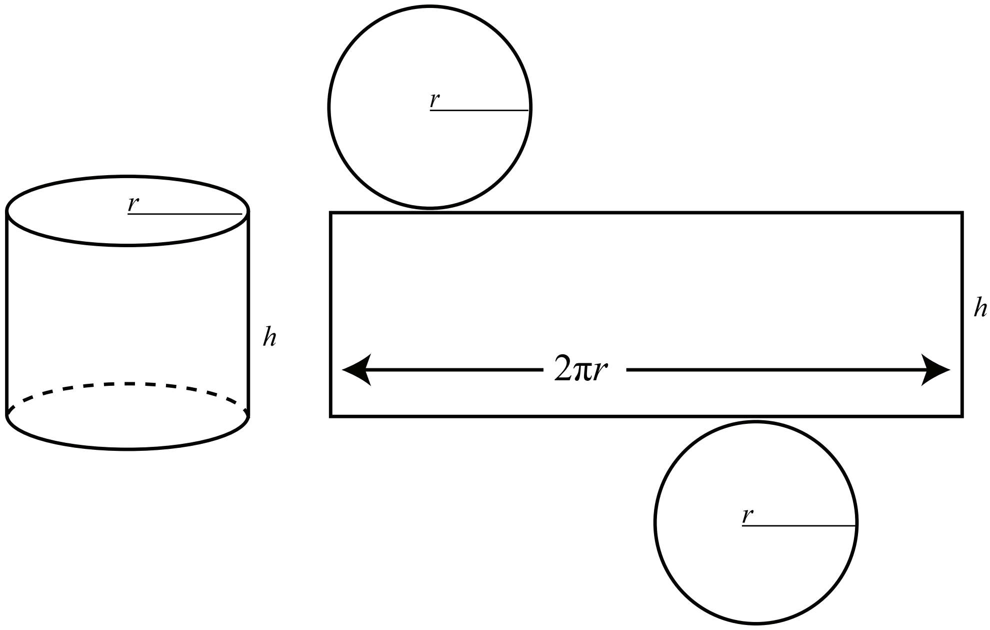




Session 3 builds on what Student Teachers know about area of both rectangles and circles to introduce the concept of surface area.

Up to now the emphasis has been on the measurement of two-dimensional plane figures. Surface area, however, relates to the multiple areas that cover a three-dimensional object.

This means that for cuboids Student Teachers need to consider not only length and width but also height. For cylinders they will be measuring a rectangular piece (such as the paper label on a tin) as well as a circular top and bottom.



Once again Student Teachers will derive formulae for the surface area of cuboids and cylinders after measuring and calculating the surface area of various boxes and tins. Student Teachers should come up with several different formulae to express the surface area of cuboids. This is an opportunity to discuss how the various formulae are equivalent.

It is also important that Student Teachers realize that every three-dimensional object in our world has surface area. That means not only other geometric figures such as pyramids, spheres, and dodecahedrons but also tables, cooking pots, and mangoes. To illustrate this point, ask Student Teachers to consider how many separate areas they would cover if they were painting an entire table (the top, four sides, the table's underneath, plus its four legs).

Unit 3/week 4, session 1: Geometric measurement – Circles, circumference, developing pi

What do Student Teachers need to know?

Circles have a *diameter*, the widest width from edge to edge measured through the circle's centre.

A *radius* is the measurement from the centre to the edge. It is half the length of the diameter. (Conversely, the diameter is twice the length of the radius.)

Circles have a perimeter, which is referred to as *circumference*.

Circumference can be measured directly by using flexible tools such as a tape-measure.

There is a relationship between a circle's diameter and its circumference, which is known as *pi*.

In cases where circumference cannot be measured directly, it can be calculated by using pi.

How do children think about these concepts?

Because the perimeter of polygons is typically taught before that of circles, children may think that circumference is a totally new topic. They need to understand that *circumference* is simply a specialized term for a circle's perimeter.

Textbooks usually treat circumference by giving an approximation of pi (3.14), then giving the formula pi $(d) = C$ or pi $(2r) = C$, and then finally asking students to calculate the circumference of several circles with different diameters.

However, to understand pi and how it relates to any circle, regardless of size, children need to measure circular and cylindrical objects, chart their findings, and analyse their data to come to the conclusion that pi is a constant that has a meaningful relationship to the diameter of the circle.

Children need to envision that when they measure around a three-dimensional cylinder they are actually measuring a one-dimensional length.

Children need to have a clear understanding of the difference between diameter and radius. This is because the circumference is usually expressed as a function of its diameter, while the same circle's area is usually calculated by using its radius.

What is essential to do with Student Teachers?

- Introduce the vocabulary of circles: *diameter*, *radius*, and *circumference*. But do not introduce *pi*.
- Have Student Teachers measure around cylindrical objects, chart their findings, and analyse their results to come to an approximation of pi ('three and a little bit more').
- Allow Student Teachers to develop the formula for circumference pi (d) = C.
- Help Student Teachers understand the difference between direct measurement and calculation by formula.

Activities with Student Teachers

Begin by referring to last week's focus on the area and perimeter of polygons and irregular surfaces. Note that those same concepts also apply to another type of figure: circles.

Student Teachers were asked to bring in cylindrical objects. After having mentioned the vocabulary relating to circles (*diameter, radius, circumference*), have Student Teachers work in groups of four to measure the diameter and circumference of the items they brought to class.

Provide some groups with a tape-measure. Give other groups string and a ruler.

Give each group chart paper with three columns labelled 'diameter (d)', 'Circumference (C)', and 'C divided by d'. Have each group chart their findings.

Discuss their 'C divided by d' columns. What pattern do they see? What do they remember about learning about pi in school? How does their direct measurement experiment relate to what they had been taught? How did this activity help them understand what children need to experience to understand pi?

Assignment

To be determined by the Instructor.

Unit 3/week 4, session 2: Geometric measurement – Area of circles

What do Student Teachers need to know?

Circles, like irregular figures, have areas that are expressed in square units.

The area of a circle, like those of irregular figures, can be estimated by direct measurement: laying a grid over the circle and counting the squares within.

There is a relationship between the radius of a circle, its area, and pi.

There are various visual images of the formula [pi (r^2) = Area] that can help children understand what this formula means.

How do children think about these concepts?

Having had the experience of using a grid to estimate the area of irregular shapes, children can transfer that understanding to the area of circles.

Textbooks usually give the formula for calculating the area of a circle [pi (r^2) = Area] before children understand why this is so.

Once again, the goal is for children to have first-hand experience before the formula is introduced. In this way they will know what a radius square looks like and how it is related to finding the area of a circle.

What is essential to do with Student Teachers?

- Ensure that Student Teachers understand that the area of a circle (like the area of any irregular surface), when measured by various tools or the approximation of 3.14, can only be an estimate. However, describing a circle's area in terms of pi is mathematically accurate.
- Have Student Teachers engage in the following hands-on activities that will help them make sense of the formula.
- Begin with the method of counting the number of squares on grid paper, and then move to the radius square method for approximating the area of a circle.

Activities with Student Teachers

Refer to last week's discussion for finding the area of irregular two-dimensional surfaces. Ask about how using a grid to estimate the area of an irregular surface might relate to today's task: finding the area of a circle.

Distribute centimetre grid paper and ask Student Teachers to place the point of their compass on an intersection and then draw a circle. If compasses are unavailable, Student Teachers can trace around the cylinders that were used in the previous session to draw circles. How can they estimate the area of their circle?

As you move about the room, notice how Student Teachers address the grid's partial squares.

After Student Teachers share their direct measurement strategies for estimating the area of a circle, distribute the 'radius squares' handout. Note their reaction to how r^2 can be visualized as an overlay on a circle.

Have Student Teachers develop a formula to express how pi and radius squares (or radius squared) relate to a circle's area.

If there is time, introduce the parallelogram method of estimating a circle's area.

End the session by asking Student Teachers how today's and the previous session's activities informed their understanding of how pi relates to a circle's circumference and area.

Assignment

Have Student Teachers bring a box and a cylinder to the next class session for use in calculating surface area.

Unit 3/week 4, session 3: Geometric measurement – Surface area of cuboids and cylinders

What do Student Teachers need to know?

All three-dimensional objects (not just geometric shapes such as cubes, cuboids, and cylinders) have surface area.

Surface area is the sum of all the areas covering any three-dimensional object.

A *net* is a two-dimensional representation of a three-dimensional object's surface area.

Formulae for finding the surface area of cuboids may be expressed in various ways.

How do children think about these concepts?

Just as children tend to confuse the concepts of area and perimeter, they also confuse the concepts of surface area and volume.

This may be attributed to a textbook's introduction of these two models of measuring three-dimensional objects in quick succession.

Although surface area is a measurement of a three-dimensional object, its measurement is expressed in square—not cubic—units.

It is helpful for children to envision surface area not as a geometric abstraction but as the simple act of wrapping a box with paper.

Omitting the overlap required for taping, how much paper is needed to cover the box?

For cylinders, what is the area of a tin's paper label added to the area of its circular top and bottom?

For children, the adjective *surface* in the term *surface area* can imply 'only one surface'. Thus, they may assume that they need to find only one area of the surface rather than the sum of all the object's surface areas.

In most textbooks, the section on surface area includes diagrams of cuboids and cylinders labelled with their dimensions. This makes it relatively easy for children to simply insert numbers into a surface area formula and perform the calculations. However, this is not how the surface area of boxes and cylinders is found in the real world.

What is essential to do with Student Teachers?

- Introduce surface area as a generalized concept that applies to all three-dimensional objects, not something unique to cuboids and cylinders.
- Introduce the idea of a net by cutting apart a box to demonstrate how a three-dimensional object can be transformed into a two-dimensional plane figure.
- Have Student Teachers work with boxes and tins to explore the surface area of cuboids and cylinders.
- Have Student Teachers develop multiple formulae to express equally valid ways of calculating surface area.

Activities with Student Teachers

Begin by reminding Student Teachers how they explored the area of polygons and circles, developing methods for direct measurement and mathematical formulae. Mention that they learned that even irregular figures have a perimeter and area.

Point out several examples in the classroom that have multiple surfaces and thus several areas that would need to be added to calculate surface area; e.g. four walls that need to be painted.

Display a cardboard box and discuss that it has three dimensions, not only length and width but also height. Create a net by cutting apart the box. Ask Student Teachers to discuss how an object they originally perceived as three-dimensional now has only two dimensions. Ask how they might find the area of the cardboard.

Divide Student Teachers into groups of four. Half the groups will work with a box, the other half with a cylinder. Have them use various measurement tools (ruler and tape-measure) to discover the surface area of their object.

Ask them to find a generalized formula for their experiment.

Have Student Teachers share their experiences and their formulae. If Student Teachers devise several valid formulae, show how these are equivalent. (If they have not come up with various formulae, introduce at least one other.) Ask if some formulae are more efficient than others.

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 3/week 5: Geometric measurement –Volume, square roots (surds), and right triangles

Session 1: Introduction to the volume of cuboids and cylinders
Session 2: Introduction to square root
Session 3: Introduction to the Pythagorean theorem

Faculty preparation for the upcoming week (1–2 hours)

- Look through the following websites:
  - Difference between volume and capacity:
    - http://tinyurl.com/Volume-vs-Capacity
  - Pythagorean theorem:
    - http://tinyurl.com/Alt-Pythag-Theorem
- Download and print out for Student Teacher use:
  - Geoboard dot paper:
    - http://tinyurl.com/GBoard-Dot-Paper
  - Plain dot paper:
    - http://tinyurl.com/Plain-Dot-Paper
  - Centimetre grid paper:
    - http://tinyurl.com/GridCm
  - Alternative proof of the Pythagorean theorem:
    - http://tinyurl.com/PythagProof2
- Bring to class:
  - Plain paper
  - Graph paper
  - Rulers
- Read through the plans for this week's three sessions.

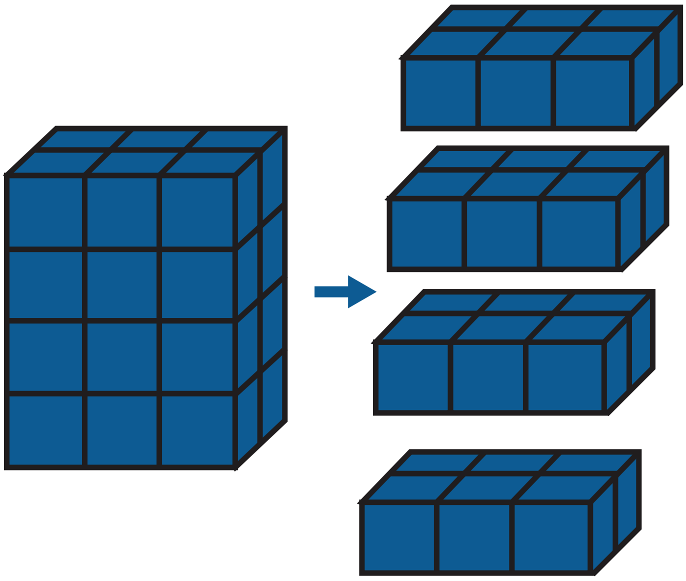
Weeklong overview

Session 1 addresses the concept of volume, noting that volume is a characteristic of all three-dimensional objects, not just the cubes, cuboids, and cylinders that will be studied in this session.

The volume of a rectangular box is expressed in cubic units, meaning how many 'unit cubes' would fill the box.

To visualize this, Student Teachers should think of covering the bottom of the box with a layer of unit cubes. This would be the area of the base. From that base, additional layers would be built until the box is full. Thus, the volume of the box would be the area of the base multiplied by the number of layers (the height in units of the box).

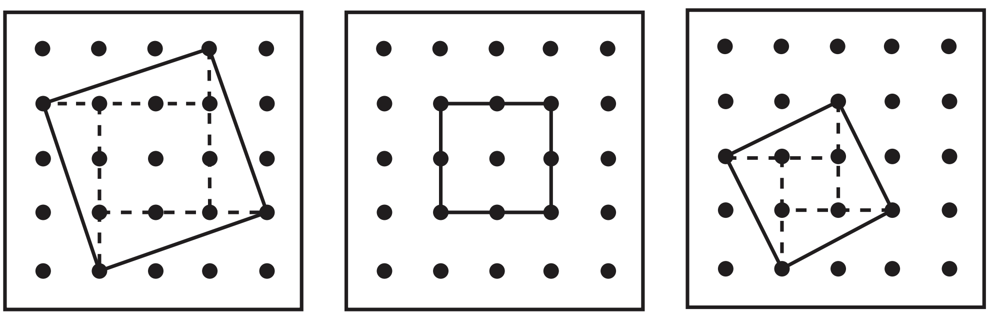
In the following illustration, the base of the cuboid is covered by six cubes. Then three more layers are added. The area of the base (6 cubic units) multiplied by height (4 layers) results in a volume of 24 cubic units.



This layering model helps Student Teachers visualize why multiplying the length, width, and height is how volume is calculated.

Student Teachers will extend this understanding of volume by finding the volume of cylinders. Although the base of a cylinder cannot be covered with a whole number of cubes, Student Teachers can apply their knowledge of finding the area of a circle and then determine how many layers of that area would fill the cylinder.

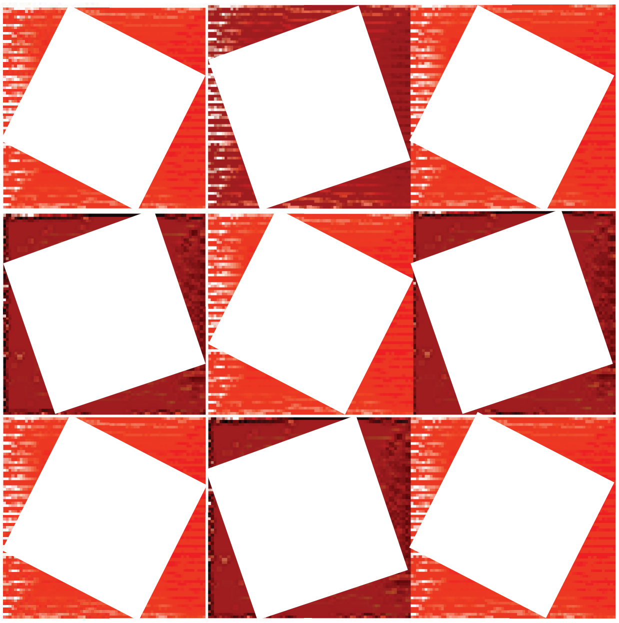
In Session 2, Student Teachers will be introduced to the concept of square root as the side of a given square. By drawing squares on 5 x 5 dot paper, Student Teachers will discover that in addition to the 1 x 1, 2 x 2, 3 x 3, and 4 x 4 'upright squares', there are also 'tilted squares'. Although the area of the tilted square can be expressed as a whole number, its side length is not a whole number.



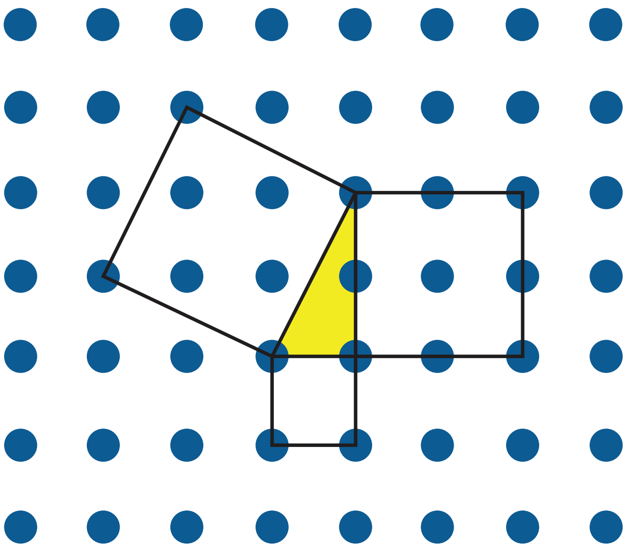
Area –10 Side > 3 Area – 4 Side – 2 Area – 5 Side > 2

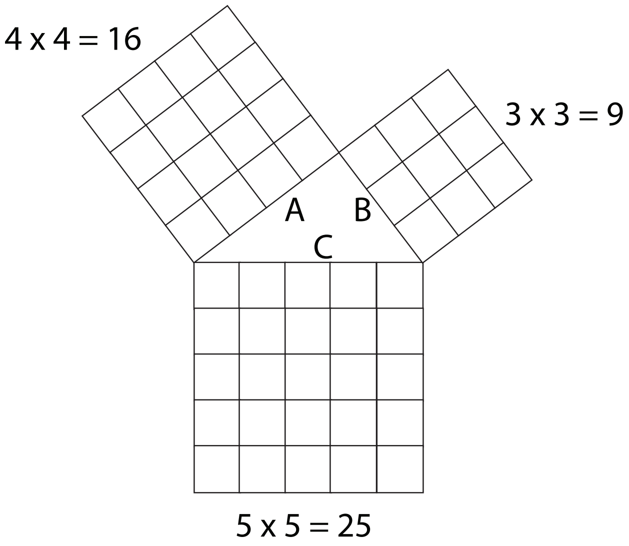
This gives rise to the concept of square root.

Notice the use of both the nine upright dark squares each surrounding the nine white tilted squares in this quilt design. This design is actually an alternate proof for the Pythagorean theorem.

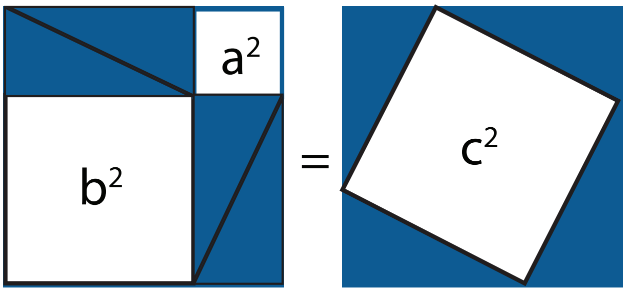


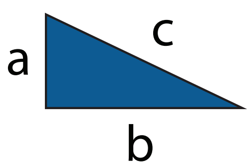
Exploring the nature of squares and square roots leads to the week's Session 3, a brief introduction to the Pythagorean theorem. Student Teachers will not be told the formula $A^2 + B^2 = C^2$. Instead, by having Student Teachers build squares on the sides of right triangles using geoboard dot paper, they will see how the hypotenuse is related to the triangle's other two sides.





Here is the proof for the quilt above:





Unit 3/week 5, session 1: Geometric measurement – Volume of cuboids and cylinders

What do Student Teachers need to know?

Volume is the amount of space taken up by a three-dimensional object.

Volume can also refer to the capacity of an object (such as an empty box or cup) that can be filled.

Volume is expressed in cubic units.

The volume of a prism or cylinder is calculated by multiplying the area of its base by its height.

How do children think about these concepts?

Children often confuse surface area and volume, both attributes of three-dimensional objects and usually taught in tandem.

Children need to work with nets to understand surface area and then fill boxes and cylinders to understand volume. In this way they can see that an empty box can be filled to illustrate volume, and the same box, flattened into a net, demonstrates surface area.

Children tend to wonder why the volume of a cylinder can be expressed in cubic units when the cylinder is round and cannot be tightly packed with unit cubes to calculate volume.

This is why it was important to stress (during the sessions on the area of irregular figures and circles) that square units can be used as a form of measurement even if the shape under consideration is not square.

The same can be true for cubic units. A three-dimensional figure does not have to be a cuboid to discuss its volume in cubic units.

Volume is something children experience every day without realizing it. Their mothers may use measuring cups when cooking; and bottles may have their volume noted on the label. These examples illustrate how an object's capacity can be expressed in cubic units even if the item is not a cuboid.

What is essential to do with Student Teachers?

- Introduce volume as a generalized concept that applies to both three-dimensional solids as well as any three-dimensional object that has a capacity that can be filled.
- Have Student Teachers use what they know about area to calculate the volume of the boxes and cylinders that they brought to class.
- Have Student Teachers develop generalized expressions that express how to calculate the volume of boxes and cylinders.

Activities with Student Teachers

Begin by reminding Student Teachers of their work finding the area of two-dimensional irregular shapes, polygons, and circles. Note that this will be a key element as they work with the three-dimensional concept of volume.

Divide Student Teachers into groups of four, giving each group both a box and a cylinder. Ask them to calculate their objects' volume in cubic units.

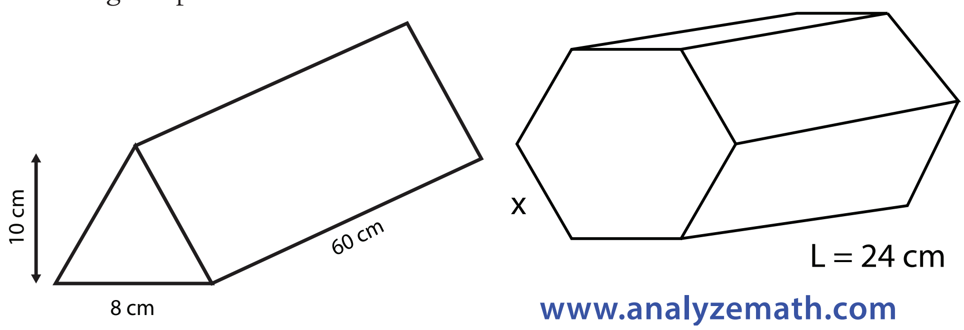
As you circulate around the room, notice if they use the area of the base as a starting point and how they incorporate the object's height into their calculations.

Have Student Teachers record their findings on chart paper and present them to their classmates.

- For cuboids: length, width, height, and volume
- For cylinders: radius, area of the base, height, and volume

As Student Teachers present their work, note if their method was to build layers upon the object's base area to find volume. Ask Student Teachers to derive a generalized formula for their work.

Finally, fold one sheet of paper into thirds and another into sixths, creating triangular and hexagonal prisms.



Ask Student Teachers how they might find the volume of these two new prisms, given what they discovered by working with rectangular boxes and cylinders. What generalities do they notice?

Assignment

Have Student Teachers visit this online tutorial about volume versus capacity:

➢ http://tinyurl.com/Volume-vs-Capacity

Unit 3/week 5, session 2: Squares, tilted squares, square roots (surds)

What do Student Teachers need to know?

The side length of some squares will be rational numbers. Other squares will have side lengths that can only be expressed as an approximation or as the square root of the square's area.

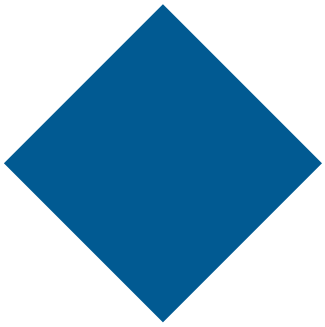
Numbers have two square roots, one positive and the other negative.

How do children think about these concepts?

When asked to draw squares on grid paper or dot paper, most children draw only upright squares. Until prompted, they do not consider 'tilted squares'.

This is similar to a very young child's perception of a triangle, not seeing a three-sided figure as a triangle unless it is oriented with its base 'down'.

Even older children may look at a tilted square and call it a *diamond*, not perceiving that it has all the characteristics of a square.



When asked to name the side length of a square with an area of two, children usually say 'one'. This is because they do not have the words to describe an irrational number.

When looking at a square cut in half along its diagonal, children often incorrectly say that this results in 'two equilateral right triangles' with sides of 1, 1, and 1.

They do not perceive that the hypotenuse is longer than the other two sides.

What is essential to do with Student Teachers?

- Have Student Teachers use dot paper grids to draw both upright and tilted squares.
- Introduce the concept and notation of square root.
- Introduce the idea that the square roots of certain numbers are irrational numbers and can only be expressed as approximations.

Activities with Student Teachers

Distribute dot paper that is arranged in five-dot squares. Have Student Teachers draw as many different-sized squares as possible, having them label the area of each. (The upright squares will have areas of 1, 4, 9, and 16. The tilted squares will have areas of 2, 5, 8, and 10.)

Note which Student Teachers stop at 1, 4, 9, and 16 and think they have discovered all possibilities. Prompt them to go beyond these, letting them know there are several more squares to discover. (Do not be surprised if they draw the same 1, 4, and 9 squares in different places on their dot paper.)

Begin by discussing the area and side lengths of the upright squares, noting that the area of a square is found by multiplying its side lengths, and this side length is called the *square root* (or *surd*) of a given square.

Ask for the square root of each of the upright squares and introduce the format $\sqrt{16}$ as a way of expressing 'the square root of 16'. This can be thought of as '4 is the side length of a square with an area of 16'.

Next, turn Student Teachers' attention to the tilted squares they drew. Ask them to consider the square with an area of 2. What is its side length? What number multiplied by itself equals 2? If Student Teachers have calculators, they may try to find this number by guess-and-check, coming up with an approximation of 1.41.

However, this is not exactly the square root of 2. In fact, the most accurate way to express the square root of 2 is to write it as $\sqrt{2}$. Following this line of thought, ask for the side lengths (square roots) of the other tilted squares.

Mention that $\sqrt{2}$ is called an *irrational number*, because it cannot be written as a terminating or repeating decimal. Remind Student Teachers that fractions as well as terminating and repeating decimal numbers are called *rational* numbers.

You may want to note that pi is also an irrational number, although for practical purposes when measuring circular objects in the real world we tend to use pi's approximation of 3.14. (Mention that computers have calculated pi to over 2577 billion decimal places—and $\sqrt{2}$ to over one million decimal places—without finding a repeating pattern of digits.)

Remind Student Teachers of their work with the multiplication of integers, and ask which factors could produce the whole number of four. As both 2 x 2 and -2 x -2 equal 4, 4 can be said to have two square roots, one positive (2) and the other negative (-2). In fact, every positive number has two square roots. If Student Teachers ask about the square roots of negative numbers, briefly mention that these are called *imaginary numbers*.

Assignment

To be determined by the Instructor.

Unit 3/week 5, session 3: Introduction to the Pythagorean theorem

What do Student Teachers need to know?

Right triangles have a base, height, and hypotenuse.

A right triangle's hypotenuse can be calculated by using the squares of its base and height. This is known as the Pythagorean theorem.

There are certain right triangles whose base, height, and hypotenuse are whole numbers (or multiples of whole numbers).

How do children think about these concepts?

Many adults remember 'A squared + B squared = C squared' from secondary school geometry. However, few adults know why this equation makes sense. Thus, when children are given this formula without having opportunities to explore proofs of the theorem, they have little understanding of it.

Even if children have seen a visual proof of the Pythagorean theorem (such as squares being drawn on each side of a triangle), they are unaware that there are literally hundreds of proofs for the theorem. It is mathematically important that students realize a theorem can be proved in multiple ways.

Children may overgeneralize and assume that if the Pythagorean theorem applies to all *right* triangles, then it must apply to all triangles. Having children explore this will help them understand the power of a counterexample.

What is essential to do with Student Teachers?

- Have Student Teachers use what they learned about building upright and tilted squares and the distance between points on a grid to build squares on the sides of right triangles.
- Have Student Teachers notice the additive relationship among the squares they drew on the sides of right triangles.
- Have Student Teachers devise the formula for the Pythagorean theorem.

Activities with Student Teachers

Remind Student Teachers that they were able to draw a 'tilted square' with an area of 2, 5, 8, and 10 units in the prior session.

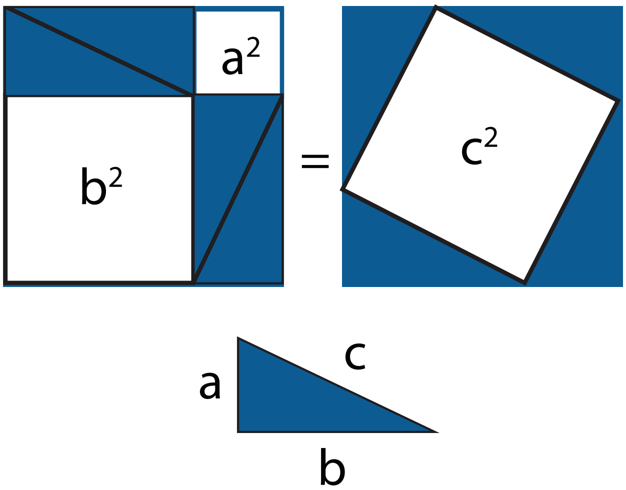
Distribute dot paper and ask Student Teachers to draw a right triangle with a base of 1 and a height of 1. Then direct them to draw a square on each side of the triangle. What do they notice about the three squares that they drew? What is each square's area? Is there a relationship between the three numbers? How does this relate to one of the tilted squares they drew in the previous session?

Have Student Teachers draw a right triangle with a base of 3 and a height of 4, then draw squares on each side. Ask if they notice a pattern between their drawings. What is it? How can it be expressed using the terminology about squares and square roots presented in the previous session?

Suggest using symbolic notation for their discovery: that the area of the square on the base plus the area of the square on the height equals the area of the square drawn on the hypotenuse. Note that while we could write this as b (base) squared + h (height) squared = H (hypotenuse) squared, it is customary to label the base 'B', the height 'A', and the hypotenuse 'C', leading to the formula A squared + B squared = C squared, or $A^2 + B^2 = C^2$.

Note that Student Teachers can use this formula along with what was learned in the 'Numbers and Operations' unit about number families: you can find any one of the three terms by using the two other terms (e.g. C 2 – B2 = A2).

Tell Student Teachers that this method of using squares drawn on a triangle's sides is only one method proving the Pythagorean theorem. Distribute the handout on alternative proof of the Pythagorean theorem (http://tinyurl.com/PythagProof2) showing this diagram and ask them to think through this particular proof.



Ask Student Teachers if they think the Pythagorean theorem works for all triangles, not just right triangles. How might they test this conjecture?

Assignment

To be determined by the Instructor.

UNIT 4

INFORMATION HANDLING

FACULTY NOTES

Unit 4/week 1: Data in the elementary grades

Session 1: The data process, reading displays of data (information design), numerical versus categorical data
Session 2: Displaying data in the elementary grades: Tally marks, pictographs, line plots, bar graphs; the shape of the data
Session 3: Displaying data in the elementary grades: Scatter plots, line graphs; interpreting data

Faculty preparation for the upcoming week (1–2 hours)

- Look through the following webpages that address graphical displays of data:
  - From line plots to bar graphs:
    - http://tinyurl.com/Data-Org-Rep
  - Overall graphs:
    - http://tinyurl.com/Menu-of-Graphs
  - Misleading graphs:
    - http://tinyurl.com/Misleading-Graphs
  - Create-a-graph applet:
    - http://tinyurl.com/Create-A-Graph-Applet
  - Interactive graph applet:
    - http://tinyurl.com/Graph-Applet
  - Bar graph investigations:
    - http://tinyurl.com/Bar-Graph-Investigate
  - Comparing bar graphs:
    - http://tinyurl.com/Bar-Graph-Compare
  - Islamabad weather reports:
    - http://tinyurl.com/Weather-Images
- Download and print out for Student Teacher use:
  - Bar graph paper:
    - http://tinyurl.com/Bar-Gr-Paper
  - Centimetre grid paper:
    - http://tinyurl.com/GridCm
  - A colour copy of 'Graph Analysis 1' (one copy per every two Student Teachers) (available as a resource in the Course Guide)
  - A colour copy of 'Graph Analysis 2' (one copy per every two Student Teachers) (available as a resource in the Course Guide)
- Bring to class:
  - Chart paper
  - Chart paper markers
  - Graph paper
  - Sticky notes
  - Crayons
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 will introduce the data process, which is essentially a research process:

1) Posing a question
2) Determining the data needed to answer the question
3) Creating a data collection plan
4) Collecting the data
5) Organizing the data
6) Displaying the data
7) Interpreting the data

To prepare for the remainder of the activities in this unit, Student Teachers will engage in an activity that will allow them to move through all seven steps of the data protocol during this first class session.

Student Teachers also will:

- look at misleading graphs that serve as warnings of what not to do when representing data
- differentiate between categorical and numerical data
- consider other forms of data displays, such as weather maps and dynamic electronic information displays (e.g. a heart monitor) that use mathematical data to provide information quickly and in real time.

Session 2 is devoted to basic displays of data (tally marks, pictographs, line plots, and bar graphs) that primary grade students can be expected to develop from their own questions and experience. When interpreting these graphs, Student Teachers will be introduced to the idea that there may be a shape or pattern to the data being displayed.

Session 3 will address two types of graphs (scatter plot and line graph) that children in upper primary and lower secondary grades can create when given a dataset. Both scatter plots and line graphs will require Student Teachers to plot points on a coordinate plane as they did in the 'Algebra' unit. An informal introduction to line of best fit will ask Student Teachers to notice and interpret a trend in a scatter plot.

Unit 4/week 1, session 1: The data process, categorical versus numerical data

What do Student Teachers need to know?

Data can be represented in a variety of ways.

When collecting data, there needs to be an agreed-upon protocol and process.

Organizing raw data is a crucial step in determining how it will be represented.

Creating displays of data is a means to an end. The display is simply a visual convenience so the data can be more easily interpreted.

There are two basic types of data: categorical and numerical.

How do children think about these concepts?

Even very young children can collect data on a routine basis. This can be something as simple as posting a weather icon and the morning temperature on a calendar.

As children grow older and begin to work more formally with data, they need to read data from existing charts and graphs. This sets the stage for their understanding that the fundamental role of a data display is to help them interpret information and then make decisions on the basis of the data.

Textbooks usually spend a disproportionate amount of time having children create graphs and not enough time helping them interpret data displays. To rectify this, teachers need to be alert to graphs and charts in the media that can be brought into class for children's discussion and analysis.

What is essential to do with Student Teachers?

- Introduce the data process, emphasizing that it is really a research process.
- Have Student Teachers work though the seven-step data process to create a bar graph showing their favourite subjects in secondary school.
- Introduce the concepts of categorical and numerical data.
- Have Student Teachers consider the implications of the data they collected.

Activities with Student Teachers

To prepare for the rest of the activities in this unit, Student Teachers will engage in an activity that will allow them to move through all seven steps of the data process during this first class session:

1) Posing a question
2) Determining the data needed to answer the question
3) Creating a data collection plan
4) Collecting the data
5) Organizing the data
6) Displaying the data
7) Interpreting the data

Student Teachers also will 1) look at misleading graphs that serve as warnings of what not to do when representing data, 2) learn to differentiate between categorical and numerical data, and 3) consider other forms of data displays, such as weather maps and dynamic electronic information displays (e.g. a heart monitor) that use mathematical data to provide information quickly and in real time.

Have Student Teachers sit in four- to six-person groups for this activity.

1) Pose the following two questions:
   a. What displays of data do you see in everyday life, such as in newspapers, on the Internet, or in the doctor's office?

      Have them respond with a quick show of hands to share their experiences.

   b. As a future teacher, what is your level of confidence for teaching all the subjects required by the National Curriculum?

      This time, allow Student Teachers to reflect on this question individually for only a minute or so without asking for comments. This question will be revisited at the end of class and is intended as a reflective introduction to the data process.

2) Determine the data needed to answer the following survey question: 'When you were in secondary school, what was your favourite subject: literature, mathematics, social studies, science, or something else?'

3) Create a data collection plan: have Student Teachers raise their hands as you ask about their favourite subject among the five categories.

   How did Student Teachers feel about this informal method of data collection? Did it happen too quickly? Did they note the number of Student Teachers who favoured each topic? Were there some Student Teachers who liked more than one subject? How could data be collected—on paper—to reflect what just happened?

4) Collect the data: ask Student Teachers in their groups to note their favourite subjects and put that information on paper.

   Do not tell Student Teachers how to do this, but observe how they complete the task.

   Did they use numbers? Tally marks? Another graphic model? Did they consider it important to note each other's names? If there were Student Teachers who enjoyed more than one subject, how did the group members resolve that issue?

5) Organize the data: bring the class back together and ask how the data collected from multiple groups can be organized to create a class profile.

   Have each group share the way they collected their data. Frequently groups will use several different methods, which can emphasize that if data are to be combined from several sources, there must be a uniform data collection system.

   Once they discover how other groups handled the information, what do they suggest for aggregating their data? Once they have come to agreement, combine the data.

6) Create a data display: ask Student Teachers to confer in their small groups regarding how the whole-class data could be displayed. Some may want to create a tally chart. Others may suggest a bar graph, or a circle graph, based on percentages. (If someone suggests a line graph, which is what children might do, just note the idea at this time.) Mention that these types of displays will be discussed in the next two class sessions.

7) At this point, ask Student Teachers to think about the audience for their information. Is it only for themselves, as an item of interest? Or might their data be combined with those from other college and university classes to inform educators at the national level? How does considering their audience influence their vision of how their data might look?

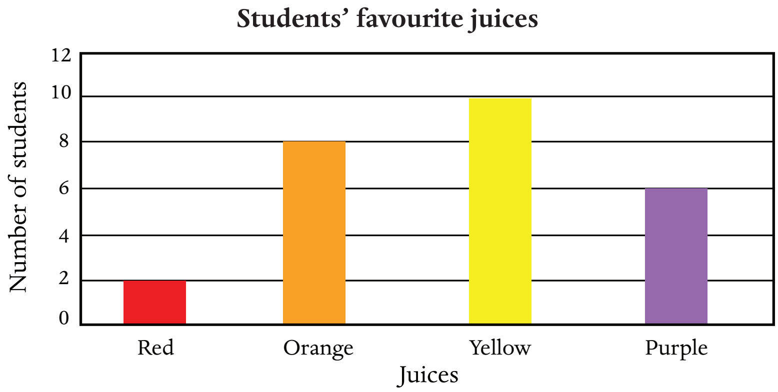
   After noting their ideas, say that you would like them to use tally marks to organize the whole-class data and then create a bar graph.

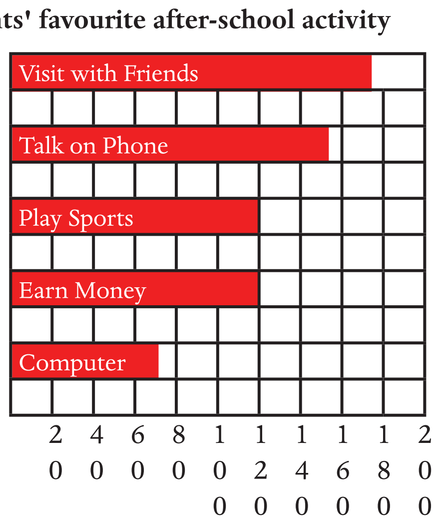
This will give you the opportunity to discuss the difference between categorical and numerical data. Mention that when creating a bar graph they will need to note the categories (subject areas) on one axis, and a number (those who favoured a particular category) on the other.

The difference between categorical and numerical data may be new to Student Teachers, so take time to help them understand that categorical data can be thought of as words, such as months of the year for Student Teachers' birthdays, the kinds of foods people prefer, etc. For this activity, the category is 'favourite secondary school subjects'.

Numerical data involves numbers, such as how many siblings Student Teachers have, their height, weight, time spent commuting to class, etc.

It is likely that Student Teachers are familiar with bar graphs, but ask half the class to create their graph with the categories on the horizontal axis and the other half with the categories on the vertical axis. These are both formats seen in graphs in news articles and reports. However, many teachers only have children work with the 'bars upright' model.





After Student Teachers have created their graphs on centimetre grid paper, ask them to consider the implications of their data. What trends do they notice? How might the information be useful to you as their Instructor? What does the data imply for their becoming teachers of mathematics?

Finally, ask how this process related to the question you initially posed: 'As a future teacher, what is your confidence level for teaching all the subjects required by the National Curriculum?'

Assignment

Have Student Teachers look at this weather site. How many different aspects of weather are represented on the page? Is the communication clear?

- http://tinyurl.com/Weather-Images

Unit 4/week 1, session 2: Graphing in the primary grades

What do Student Teachers need to know?

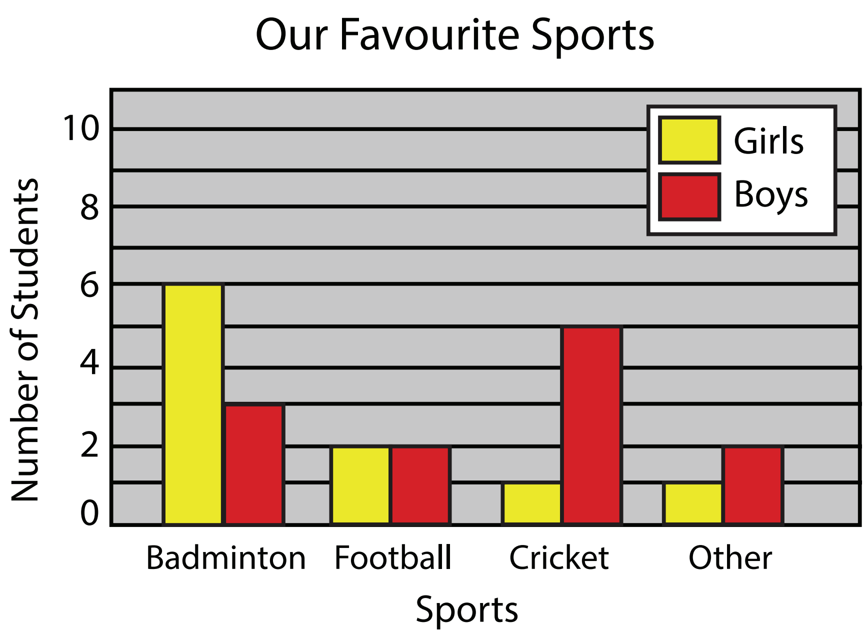
Tally marks, pictographs, bar graphs, and line plots can only be used to display discrete, not continuous, data.

Tally marks are a simple way to record information visually and then assign a numerical value to the tallies. Each item counted receives one mark; the cross mark needs to be included in the count so that each set of tallies equals five.

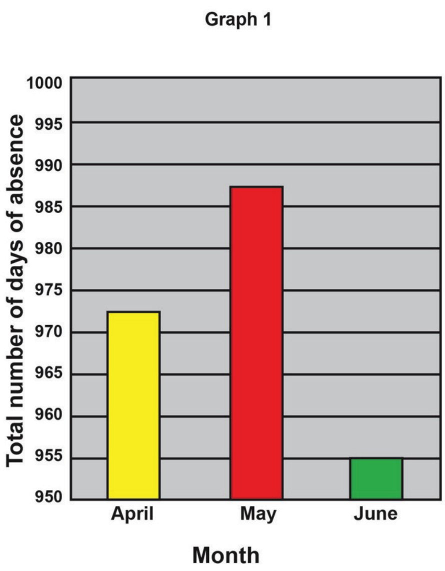
Each icon used in a pictograph might not represent a single count. Rather, the icons usually refer to multiples of numbers, such as 5, 10, or 100. Because of this, there must be a *key* to accompany the graph indicating how many units each icon represents.

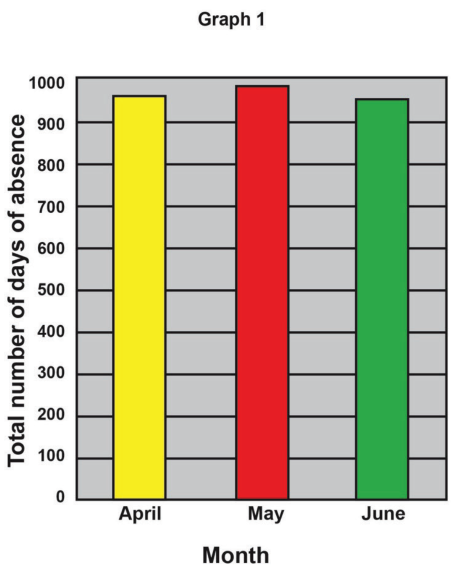
If, for example, the key is 1 icon per 100 count, a half-icon can be used to denote 50. Also, if multiple icons are used, they should be the same size to make the graph easy to interpret.

Bar graphs are a visually strong communication device when kept simple. They also can be used to show two datasets side by side, as in this illustration.



A problem with bar graphs, however, is that they are often truncated. When the bars do not begin at 0, the height difference between the bars can be misinterpreted, as in this illustration.





Line plots are based on a segment of the number line, with an 'x' written above a point on the number line for each data entry. Although they are best used for a limited range of data, they quickly show the *range*, *minimum*, *maximum*, *clusters*, *outliers*, etc. (These are all vocabulary words that students need to know.) Line plots also begin to help students notice the shape of the data.

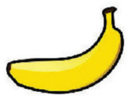
After data have been collected, the information needs to be organized before a display can be designed.

Some types of graphs are more suitable than others for conveying information.

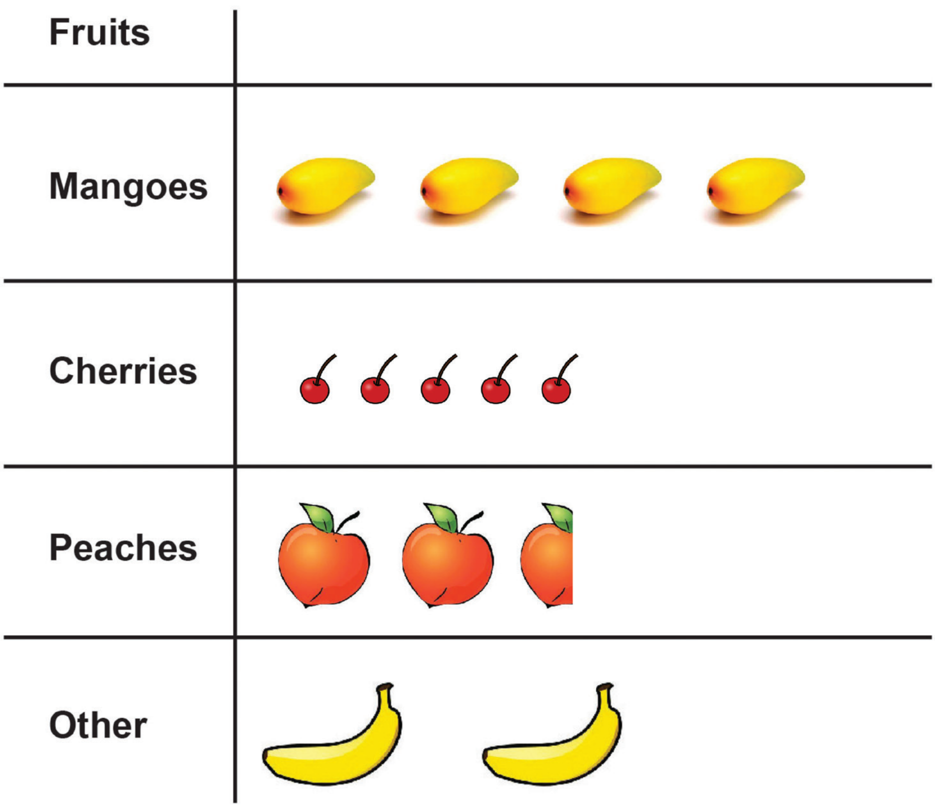
These simple types of graphs involving discrete data can provide answers to counting questions, such as, 'What was the most popular …?', 'How many people chose …?', and 'Were some things chosen the same number of times?'

How do children think about these concepts?

Children can begin creating displays of data by using tally marks to record information. Although tally marks can be jotted down informally, children need to see how tally marks can be organized into a more formal display of data, such as this chart about favourite fruit (which, if plotted on a bar graph, would display categorical data).

Favourite Fruits		
Fruit	**Tally Marks**	**Total**
	𝍸 𝍸	10
	𝍸	5
	𝍸 |	6

Pictographs are helpful as an introduction to data display for young children. However, consider the following image, which again involves fruit.



What does the half-image imply? How many respondents might there have been? Might there have been respondents who said they had more than one favourite fruit? Does the size of one fruit's image as compared to the others help or confuse the data? These are all essential information-handling issues that children need help decoding.

As mentioned in the previous session, bar graphs can be developed with the bars arranged either vertically or horizontally. Children need to see both formats.

Line plots are a simple way of displaying data, but they are not regularly used in the media. Hence, there are limited models for teachers to bring to class for students to interpret.

However, because of their simplicity of display and ability to show trends, line plots are important in the early grades, as they can be used to create bar graphs and are a way to visualize measures of central tendency, the focus of next week's sessions.

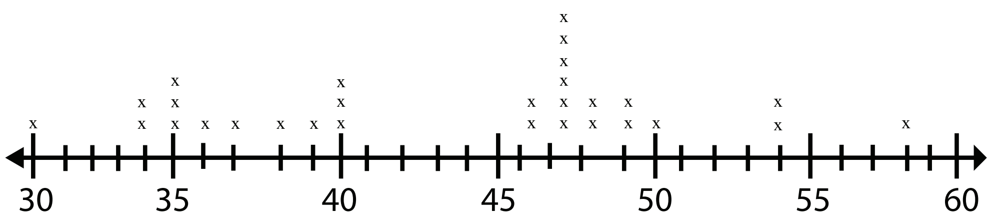


Figure: Line plot

What is essential to do with Student Teachers?

- Introduce tally charts, pictographs, bar graphs, and line plots as ways to display discrete data.
- Have Student Teachers read, discuss, and analyse samples of these types of graphs.
- Have Student Teachers construct a line plot.
- Discuss the shape of the data as displayed in various graphs.
- Discuss both the questions that graphs can answer and also the questions they might raise.

Activities with Student Teachers

Begin by reminding Student Teachers of what they learned in the 'Algebra' unit about discrete versus continuous graphs. Note that the four types of graphs being studied today are all designed to display discrete data.

Have Student Teachers recall how they used tally marks to record data and create a bar graph in the last class session. Note that in the previous session the emphasis was on constructing a graph, but that it is equally important to be able to read and analyse a graph and raise questions about what the data imply.

Without giving any further directions or comments, have Student Teachers work in pairs to discuss 'Graph Analysis 1'. What comments and questions do you notice as you circulate about the room? Call the class together for a whole group discussion of their thoughts.

On the pictograph, did they notice the key and the distortion caused by the varying size of the icons? How many responses were represented on this graph? Could this be a chart about pets? Or about favourite animals? How might 125 horses be represented?

For the bar charts, ask if they thought the double bar graphs were clear. How did colour contribute to communication?

Line plots may be a new type of display to some Student Teachers. Ask them what the data might represent. How does the lack of a title or labels affect their understanding of what the graph means? What do they notice about the shape of the data? If Student Teachers do not use terms such as *range, minimum, maximum, outlier, cluster, gaps,* etc., be sure to use them in your own description of the graph. Did they notice that the base of the graph was a segment of a number line? Ask if the data being shown was categorical or numerical.

While still working with the whole group, begin the following activity, which will result in the creation of a line plot. Ask the Student Teachers to count the number of letters in their first and last names. To illustrate how raw data needs to be organized, poll students individually (rather than have them raise their hands if they had a certain number of letters in their names). This should result in a disorganized list of data.

How do Student Teachers suggest the data be organized to translate them into a line plot? Once this has been determined, ask what they would suggest for the range of numbers displayed on the number line segment. Should they begin at 0 or some other number? How far do they need to go? How far would make a good display?

Draw the number line segment they suggested on chart paper. Then have each Student Teacher come up and put an 'x' atop their number. Do Student Teachers realize that in order for the line plot to display data accurately, all their x's need to be the same size? Have Student Teachers copy the line plot into their notebooks. Once the graph is complete, begin asking questions about its range, shape, etc.

End the session by asking Student Teachers how they might use their line plot to create a bar graph. If time allows, have them translate the line plot into a bar graph. Otherwise have them do this as a homework assignment. Ask which graph (the line plot or bar graph) they think more clearly shows the data.

Assignment

Have Student Teachers create a bar graph from the name-length line plot.

Have Student Teachers look through these two websites on bar graphs:

- Bar graph investigations:
  - ➢ http://tinyurl.com/Bar-Graph-Investigate
- Comparing columns on a bar graph:
  - ➢ http://tinyurl.com/Bar-Graph-Compare

Unit 4/week 1, session 3: Graphing in the elementary grades

What do Student Teachers need to know?

Scatter plots are created on a coordinate plane, with data points plotted on the grid. Like a line plot, they show minimum and maximum points, clusters, and outliers. Scatter plots can be used for larger datasets than line plots, and a trend line can be used to show relationships between data.

Line graphs, as noted in the 'Algebra' unit, are also composed of points plotted on a coordinate grid. However, because the data are continuous over an interval, the points are connected. An example of this would be distance travelled at a constant rate over time, growth of a plant (again over time), and the time of sunrise over several days.

Because many line graphs involve the aspect of time, they have a predictive quality: what will happen after (or between) the graph's data? This can be an opportunity to introduce the idea of extrapolation and interpolation.

How do children think about these concepts?

Scatter plots require that children are familiar with plotting points on a coordinate plane. Even if they do not have a formal (x, y) mindset, they can be taught to plot the points from the data by using the axes labels and a 1-to-1 scale on the x and y axes.

When looking at the points on a scatter plot, children soon become aware that joining points by a line on this type of graph is not warranted. This is an important step in their understanding the difference between discrete data and continuous data (which can be portrayed on a line graph, described below).

Children may think that a line graph needs to be a single straight line, as would occur for a linear function. They may not be able to interpret a line graph composed of a series of line segments even if each line segment does not represent consistent change over a given interval. This is seen in the graph below, where the data points are given for the end of the season. But no games take place between seasons. In this case, the line segments on the graph attempt to indicate trends informally, versus the way a formal regression line on a scatter plot would.

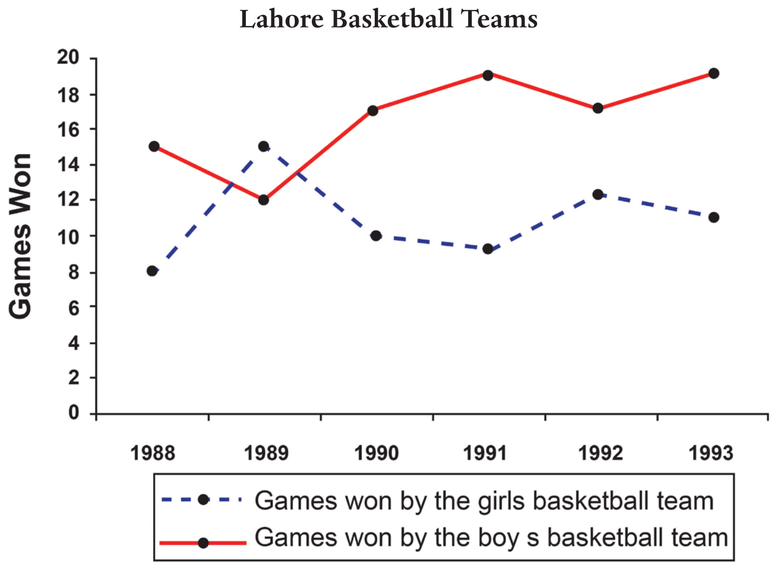


Figure: Line graph

Just as children learned to read and interpret double bar graphs, they need to do the same for multiple line graphs on the same grid, such as the above image. It is important that they note the key (in this cases two colours and a straight versus a broken line) to understand the two different datasets displayed.

What is essential to do with Student Teachers?

- Introduce scatter plots and line graphs by relating them to Student Teachers' prior work plotting points on a coordinate plane in the 'Algebra' unit.
- Have Student Teachers read, analyse, and interpret line graphs and scatter plots.
- Introduce the idea of a trend line and the concept of correlation.
- Help Student Teachers distinguish between graphs that are developed to show algebraic functions and those that are designed to communicate information to general audiences.

Activities with Student Teachers

This session begins by having Student Teachers work in pairs to read and interpret the line graph and scatter plot on the handout 'Graph Analysis 2'. Because these are both coordinate graphs, Student Teachers should be familiar with how to plot points on a coordinate grid and to read a coordinate graph, even when the grid is not visible.

After Student Teachers have worked in pairs to discuss the line graph and scatter plot, have a whole-class discussion about what they discovered, both their questions and the questions at the end of the handout. In particular, note new questions that had not suggested before.

If no one has mentioned it, take the opportunity to point out that although the weather graph is called a line graph, it is not a graph of a single linear function. This is a graph displaying observational data from real-world sources in a user-friendly manner for a general audience.

Note, too, that line graphs suggest predictions. This predictive quality allows for extrapolation (what might happen next?) and interpolation (what may have happened between data points?).

Emphasize that line graphs designed for a general audience usually connect data points that do not indicate mathematically consistent change over a given interval. Therefore, the line graphs being studied today fall into the area of practical information handling rather than algebra.

When Student Teachers consider the scatter plot of plant growth, continue the class discussion in the same inquisitive manner as above, asking: what questions did they think the graph answered? Did the graph answer the student's questions that were included on the handout? Which of these students' questions did they find intriguing? Did the answers pose any new questions? How did they decide where to draw a trend line? Does this scatter plot show discrete or continuous data?

Assignment

To be determined by the Instructor.

FACULTY NOTES

Unit 4/week 2: Measures of central tendency

Session 1: Measures of central tendency I – Frequency tables, range, mode, median
Session 2: Measures of central tendency II – Mean, selecting a model
Session 3: Measures of central tendency III – End-of-course reflection

Faculty preparation for the upcoming week (1–2 hours)

- Read the following article and look through the following websites that address measures of central tendency:
  - 'Teaching the Mean Meaningfully':
    - http://tinyurl.com/Mean-Distribution
  - 'Finding the Mean: Distribution Method':
    - http://tinyurl.com/Mean-Distrib-Method
- Download and print out for Student Teacher use:
  - Bar graph paper:
    - http://tinyurl.com/Bar-Gr-Paper
  - 'Frequency Tables and Mean' handout (available as a resource in the Course Guide)
  - 'Frequency Tables and Mode' handout (available as a resource in the Course Guide)
  - 'End-of-Course Reflection' (available as a resource in the Course Guide)
- Bring to class:
  - Chart-sized graph paper
  - Scissors
  - Enough large beans so that each group of 4 has 24 beans
  - Crayons, coloured pencils, or markers
- Read through the plans for this week's three sessions.

Weeklong overview

Session 1 begins by revisiting the work Student Teachers did with line plots (recording the letters in their names) in order to discuss frequency tables and the range of the data. Note that when they created their original bar graph from categorical data in the first class session of the unit, the order of the bars was not important.

However, when dealing with numerical data on a line plot, the resulting bar graph should show the shape of the data. They also will consider datasets where the median is not a number in the dataset, which will lead them to ask if the median of a dataset of whole numbers needs to be a whole number.

Session 2 will address the measure of central tendency that most Student Teachers will probably recall: the *arithmetic mean*, or what tends to be called the *average*. Although calculating the mean usually means doing division, there is a manipulative, more visual model that Student Teachers can use to explore this topic.

This will be an opportunity to explain that *mean*, *median*, and *mode* are all types of averages because they denote the middle of a dataset. Depending on the data's context, one of these measures may be more useful than the others when communicating data. This is why statistics (both in research and in the media) need to be considered carefully, with the reader noting which measure of central tendency the writer is using.

Session 3 is the last session of the entire course. This will be a time to reflect on both the mathematical content Student Teachers learned and the way that content was presented. There will be a handout on which Student Teachers will write their thoughts. Then, because group work and whole-class discussions have been emphasized in this course, the reflection process will be 1) individual written reflection, 2) small group discussion, and 3) whole-class discussion.

Unit 4/week 2, session 1: Measures of central tendency I – Frequency tables, range, mode, median

What do Student Teachers need to know?

Frequency charts show the same data as a horizontal line plot, only in vertical tabular form. The numbers on a line plot's number line segment are arranged in order in the left-hand column, while the number of x's on the line plot is listed in the right-hand column as tallies. The line plot and the frequency table below show the same distribution of data.

Note that a frequency table may include not just single numbers in order, but ordered intervals of numbers, as in the second table below.

Serve	Tally	Frequency
1	|	1
2	|	1
3	|||	3
4	|	1
5	||||	4
6	卌	5
7	卌 |	6
8	卌	5
9	|||	3
10	|	1

Class interval	Tally	Frequency
0–39	|	1
40–79	卌	5
80–119	卌 卌 ||	12
120–159	卌 |||	8
160–199	||||	4
200–239	|	1
	Sum =	31

Figure: Frequency table

Note the following frequency table, which makes the mistake of having the same number (20 and other multiples of 10) in two different intervals.

Classes	Frequency	Cumulative frequency
0 – 10	1	1
10 – 20	4	5
20 – 30	3	8
30 – 40	7	15
40 – 50	7	22
50 – 60	7	29
60 – 70	1	30
Total	30	

There are several commonly used measures of central tendency, each of which can be considered an average because it denotes the middle of a dataset.

The *range* of the data is found by subtracting the lowest value from the highest value.

The *mode* is the category or number that occurs most frequently in a distribution.

Usually, when looking at a line plot or bar graph, the mode would be seen as the tallest stack in a line plot or the tallest bar in a bar graph. This does not mean that the mode has the highest value in a dataset, only that it occurs most frequently.

For example, if children among several families have the ages 5, 9, 6, 1, 4, 6, 7, 6, and 3, the mode of their ages is 6 because that is the number that occurs three times, even though the oldest child is 9. If the data were put into numerical order on a line plot, it would show 1, 3, 4, 5, 6, 6, 6, 7, and 9, with 6 having the highest frequency (or stack of x's).

There can be more than one mode. This type of distribution would be termed *bi-modal,* as in the following case where the greatest number (330) appears twice (in 1999 and 2000), while the other values appear only once.

Figure: Bi-modal distribution

The *median* is the number halfway between the minimum and maximum numbers listed in an *ordered* dataset. Thus, in the dataset given about children's ages, there are nine numbers. When ages are organized from youngest to oldest (1, 3, 4, 5, 6, 6, 6, 7, 9), the median would be 6 because half the data (four values) are to the left of the 6, whereas the other four values are to the right of that 6.

Note that in the dataset of children's ages, there was an odd number of values, which allowed for finding the median by counting off an equal number of values to the right and to the left. However, the median in a dataset with an even number of values might not be a number in the dataset. It might not even be an integer.

How do children think about these concepts?

When children create frequency tables for categorical data used to create tally charts, bar graphs, or pictographs, they do not need to design their table in ascending or descending order.

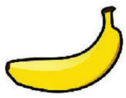
Favourite Fruits		
Fruit	**Tally Marks**	**Total**
	𝍸 𝍸	10
	𝍸	5
	𝍸 |	6

Figure: Pictograph

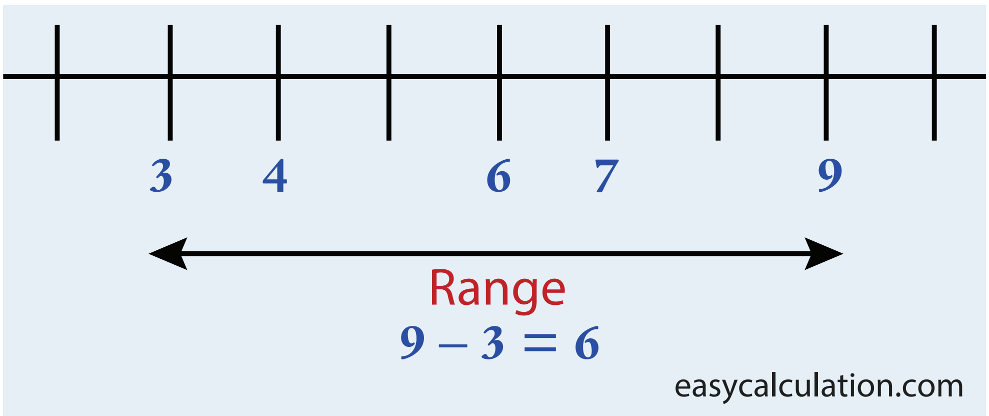
However, when working with numerical data, it is crucial for children to order their numbers in frequency tables. This will allow them to begin seeing the shape of the data as it would look in a line plot or on a bar graph.

Children may think that a frequency table can only be created with single numbers, such as the scores in the table on the left. However, sometimes (especially with large datasets) it is necessary to organize the data in consistent intervals, as in the table on the right.

Serve	Tally	Frequency
1	I	1
2	I	1
3	III	3
4	I	1
5	IIII	4
6	卌	5
7	卌 I	6
8	卌	5
9	III	3
10	I	1

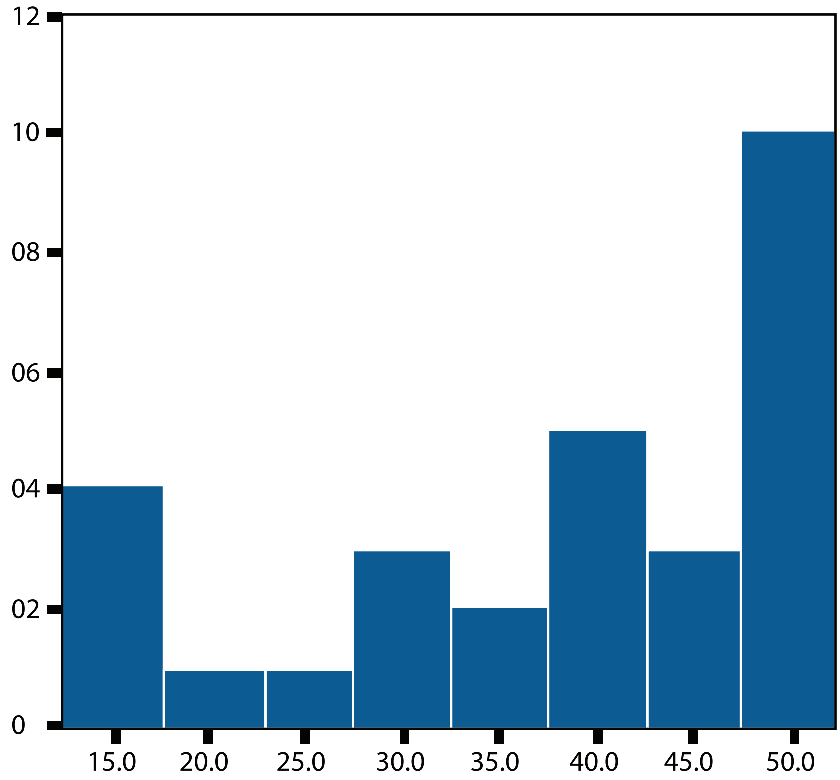
Class interval	Tally	Frequency
0–39	I	1
40–79	卌	5
80–119	卌 卌 II	12
120–159	卌 III	8
160–199	IIII	4
200–239	I	1
	Sum =	**31**

When considering the range of data, children tend to say the data 'range from a (the lowest value) to b (the greatest value)'. While this is an acceptable way to describe the range, children need to know there is another way to describe it: $(b - a)$.



When considering the mode, a clear data display allows children to have a visual model of the highest value. Confusion occurs, however, when a display of the dataset has two peaks of the same height. Children need to be assured that there can be more than one mode in a dataset.

Once children know that there can be more than one mode, they may see a graph such as the one below and become confused when they see that there are values with the same height (as in the intervals labelled 20.0 and 25.0, and 30.0 and 45.0). They may assume these are multiple modes, not remembering that the mode needs to be the most frequent in the dataset (50.0), not just the 'height of the bars that appear most frequently'.



If a dataset contains an odd number of entries, it is relatively easy to find the median. Consider this dataset: 1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 26. Children can use the 'counting from both ends' strategy to arrive at the correct median: 3.

Children may think, incorrectly, that to find the median they should add the numbers in the dataset and divide that sum by 2. When the sum for the above 11 entries (55) is divided by 2, the result (27.5) is not the median.

If a dataset contains an even number of entries, things become more complicated because there is no middle value in the dataset when counting from both ends. Instead, the median is somewhere between the middle two values.

In the dataset (3, 7, 16, 25, 32, 39), the median is somewhere between 16 and 25. The median is found by adding 16 and 25, then dividing the sum by 2, resulting in 20.5, which is neither a number in the dataset nor an integer.

Children need help knowing how to interpret their accurate calculations, especially if a whole number would be the only sensible solution to a real-world data situation. For example: one family has three children; another family has two. The median number of children per family is 2.5—surely not a realistic result, but a valid mathematical one.

What is essential to do with Student Teachers?

- Building on what Student Teachers have learned informally about data tables when creating tally charts for categorical data, introduce the necessity of putting numerical data in numerical sequence to show patterns, both numerical and visual (to show the shape of the data).
- Introduce the concept of range.
- To introduce the mode, have Student Teachers read and analyse several charts and graphs to see how the mode appears in various data displays.
- Provide opportunities for Student Teachers to ask questions about multi-modal graphs.
- Introduce the concept of the median and have Student Teachers use three methods to find the median of a dataset.
- Provide Student Teachers with opportunities to analyse situations in which the median is part of both odd- and even-numbered datasets and in which the median is not an integer.
- Note that both the mode and the median (not just the arithmetic mean) are averages that show what is typical about the dataset.

Activities with Student Teachers

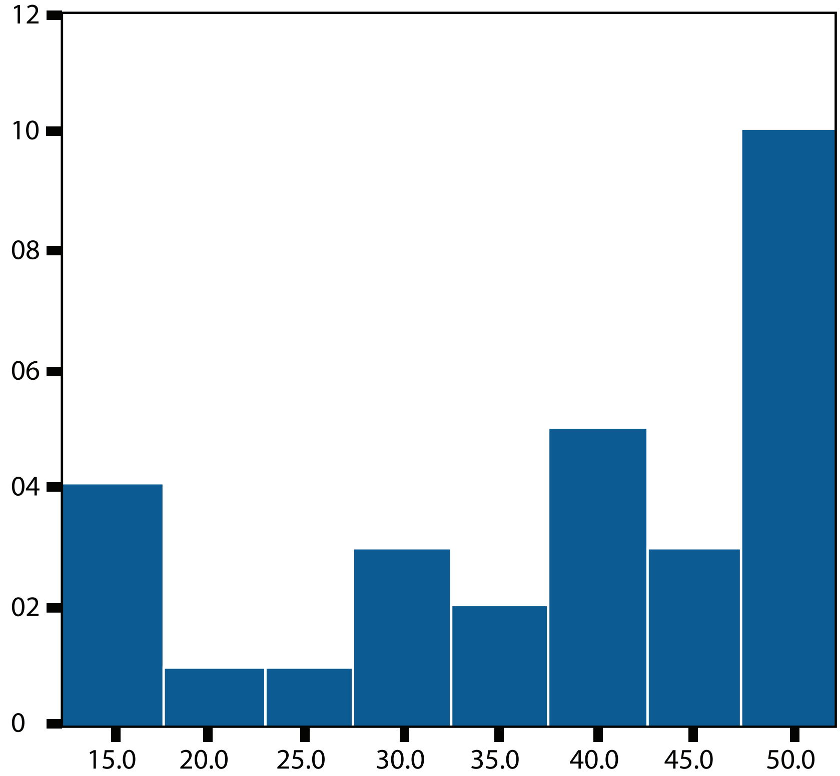
To build on what Student Teachers have learned informally about data tables when creating tally charts for categorical data and function tables in the 'Algebra' unit, use the 'Frequency Tables and Mode' handout to introduce the necessity of putting numerical data in a numerical sequence (either ascending or descending). This will allow Student Teachers to discern both numerical patterns and the shape of the data.

Ask Student Teachers how they would describe the *range* (or spread) of their name-length data that they recorded in their notebooks. Note if they use the conversational ('from … to …') format or if they suggest a formula they learned: subtract the lowest value from the highest value. Discuss how both ways of expressing the range are valid, but that one is more conversational and the other more formally mathematical.

Introduce the mode by having Student Teachers refer to the name-length line plot and bar graph they created last week. Which number of letters has the highest stack of x's in the line plot and the highest bar on the bar graph?

If there is only one answer to this question, this number is the mode. If there are more than one stack or bar with the same highest frequency, the dataset is multi-modal (bi-modal if there were two). Emphasize that the mode is the data that occur most frequently in a given dataset.

Perhaps in Student Teachers' name-length data there are two other stacks or bars of the same height—but not as high as the single tallest one, such as in the graph below.



This does not mean that the two shorter stacks (with a frequency of 1 and 3) are the modes. Although 1 and 3 each occur twice, they have a lower frequency than the highest frequency (represented by the tallest bar) at 10. In this case the mode is 10.

For a very long name, it is likely that the line plot and bar graph will show a very short stack or bar. Student Teachers may wonder why, if a name is so long, its stack or bar is so short. This is an opportunity to clarify that the height of stack or bar does not represent name length. Rather, it represents the number of people with a particular name length. This is called *frequency* because it shows how frequently a particular number occurs in the dataset.

Thus, a long 20-letter name may have a frequency (bar height) of 1, while a 12-letter name may be more typical of the dataset and occur 8 times (and have a bar height of 8).

To introduce the concept of the median, define it as the value halfway from each end of the dataset. Provide a short dataset with an odd number of entries such as 2, 2, 3, 5, 7, 8, 9.

Because there are an odd number of entries, the median can be found by counting off a pair of numbers, one from each end, until the single middle value is reached. Ask Student Teachers how they might describe the mean to children (e.g. 'half the numbers are to the left of the median, the other half are to its right').

Next, have the Student Teachers write the numbers (which are already in ascending order) in a horizontal line on centimetre grid paper, one number in each cell. Have them cut out the strip of paper and fold it in half. What do they notice? (That the crease falls on the middle number, in this case, 5.)

The point of using these two hands-on methods is to give Student Teachers a direct experience in understanding the median. However, with larger datasets, counting from each end could cause errors, and a strip of paper that is perhaps 43 cells long would be unwieldy.

Now that they have an understanding of the median as the middle number of a dataset, it is appropriate to introduce the formula where n is the number of entries: $(n + 1)/2$. In the case of the dataset with 7 entries, this would be $(7 + 1)/2$ or $8/2$. This gives the number 4, indicating that the median is the fourth number in the dataset, which in this case is 5.

Note that all of three of the above examples referred to an odd-numbered dataset.

Ask Student Teachers what would happen if you added another entry to the set, perhaps the outlier, 20. The new dataset would be 2, 2, 3, 5, 7, 8, 9, 20. What is the median now?

Have them explore this question by using the three methods already discussed: 1) counting from the end, 2) folding the numbered paper strip, and 3) using the formula. What do they notice?

Given that the median is between two numbers, ask what they think they should do now. (This is relatively easy to do, as the number 6 is right between the 5 and the 7.) Ask if they think that the median of a dataset can be a number not in the dataset.

Challenge them to find the median in this short dataset: 8, 12, 20, 38. In this case the median is halfway between the 12 and the 20, but it is not part of a natural sequence as was 5, 6, and 7.

Have Student Teachers find the median of 2 and 5, as in the case of the median number of children per family in two families, one of which has two children and the other which has five children. In this case, the median in this even-numbered dataset will be 3.5. Ask Student Teachers to interpret this answer as it applies 1) to a real-life situation and 2) to mathematics without a context.

End the session by noting that when Student Teachers found the median for a dataset with an even number of entries, they were using a strategy that will be discussed more fully in the next session: the arithmetic mean.

Assignment

To be determined by the Instructor.

Unit 4/week 2, session, 2: Measures of central tendency II – The arithmetic mean, deciding on measures of central tendency

What do Student Teachers need to know?

The arithmetic mean is calculated by adding the values of all the items in the dataset, then dividing by the number of items. This measure of central tendency implies that each item in the dataset has the same value. Sometimes the mean is not an integer.

When calculating the arithmetic mean, teachers need to think about whether they consider this a practice activity in calculation (with sums and division being done by hand) or an activity in information handling, which would be an appropriate place for students to practise what they know about addition and division and use a basic hand-held calculator.

The arithmetic mean is often called the *average*. However, all measures of central tendency that consider what is typical of a dataset can be considered to be an average. Hence, the mode and the median should be considered averages, too.

When reading articles containing statistics, students need to be alert to which measure of central tendency the writer has chosen to use and for what purpose. Consider the following dataset:

0, 0, 0, 0, 0, 50, 50, 100, 100, 100, 4000

The mean is a useful measure of central tendency to communicate an even distribution.

However, the mean is influenced by all the data, including extremes and outliers. In the above dataset, the many 0s and the large outlier have a dramatic effect on the mean, which is 440.

If these data show salaries of 11 people in a work group (5 volunteers earning no money, 5 interns receiving a small amount [50 or 100], and a project director earning 4000), then the mean average salary for the organization would be 440. Using the mean would make it appear that the 5 persons represented by 0 had a 440 income, which is not true.

On the other hand, if the data represents 11 people contributing money to share with each other on the basis of need, each person would receive a 440 share, which would be true.

In situations where the data are skewed, as in the dataset above, the median may be a more realistic measure because it is not influenced by outliers. Thus, the median of this dataset is 50, quite different from the mean of 440. (This is why the median is often used in situations such as real estate values, because in one neighbourhood there may be many similarly priced homes and an outlier, such as a single house that is quite expensive.)

The mode in the above dataset is 0, because that is the number that occurs most frequently (5 times). If the mode were used as the measure of central tendency and this dataset related to earnings, it would show that the unemployed are the most typical segment of the dataset.

Student Teachers need to ask which measure of central tendency is most appropriate in a given context. Which 'average' is most typical of the above dataset? 440? 50? 0? There is no easy answer to this. It depends on the point of view the writer wishes to communicate, and this illustrates why a reader must note the context of the situation and the statistical choice the writer has made.

How do children think about these concepts?

Simple arithmetic means can be calculated by hand as an introduction to the concept. When children are presented with real-life data, however, the paper-and-pencil method becomes cumbersome. This is where the use of a basic hand-held calculator can not only make the task easy and quick, but also move children to think about the implications of the mean.

Just as children find it difficult to believe that the median can be a number not in the dataset (or not an integer), they can be equally confused when the arithmetic mean turns out to be a fractional or decimal number. (How can a family have 2.3 children?)

They need to understand that they probably did the calculation correctly, but the mean represents what would happen if all the data were equalized for each item in the dataset. This becomes more understandable when children have another context for their data, such as discovering their test scores resulted in a 93.2 average for the term.

The definition of the arithmetic mean as equalizing each item in a dataset can be modelled with manipulatives.

Children can learn to calculate the mode, median, and arithmetic mean as measures of central tendency. However, their procedural ability does not ensure that they understand why one measure of central tendency might be preferable to another in a particular context.

Because datasets without context become purely procedural for children, they need to work with datasets linked to a real-world situations to understand the implications of the data. (This is similar to children needing a context when interpreting relationships displayed in graphs.)

Thus, to help children interpret datasets, teachers need to find (or create) datasets that have a real-world connection to their students' lives.

A dataset that includes negative integers will have an impact on the mean.

What is essential to do with Student Teachers?

- Discuss the arithmetic mean, noting that this is the third type of average they have studied.
- Have Student Teachers use simple manipulatives to explore the concept of the arithmetic mean.
- Have Student Teachers calculate the arithmetic mean of a dataset, preferably by using a basic hand-held calculator.
- Discuss how negative numbers, extremes, and outliers can influence the mean.
- Discuss why a writer might select a particular measure of central tendency (mode, median, or mean) to communicate data to a given audience.
- Discuss the use of technology in information handling.

Activities with Student Teachers

Have Student Teachers work in pairs on Activity 1 on the 'Frequency Tables and Mean' handout to create a frequency table organizing those entries in ascending order and to find various measures of central tendency.

23, 25, 32, 32, 33, 40, 40, 41, 41, 41, 49, 49, 50, 51, 52, 58, 67, 73, 77

After Student Teachers have found the median and mode, ask how they could find the mean of the dataset. (They will probably say that they would add the numbers in the dataset and then divide that sum by 19.) This is a formulaic strategy: add all the numbers and divide by the number of entries.

This is an opportunity to discuss using available technology when working with information handling. Have Student Teachers who have a hand-held calculator with them compute the mean. Ask the remaining Student Teachers to calculate the mean of the dataset by pencil and paper. Have them raise their hands when they find the mean. Overall, which group finished first?

Ask Student Teachers to consider the goal of information-handling activities. Are they for numbers and operations practice? Or for dealing with data quickly and efficiently so that children have enough time in class to analyse the data and discuss its implications?

Student Teachers used the arithmetic mean earlier in this unit when they found the median of an even-numbered dataset by adding the two numbers in the middle of the set and dividing that sum by two.

However, being able to do this (adding and dividing either by hand or by using a calculator) does not ensure that they understand the concept of distributing 'extras' from greater values in the dataset to those entries with lesser values, thereby levelling all the entries to a common number.

Activity 2 on the 'Frequency Tables and Mean' handout will help Student Teachers visualize how the 'distributing and levelling' process works to find the arithmetic mean.

Tell Student Teachers that they will be using the dataset (2, 3, 3, 4, 6, 6), which represents the number of people living in six different families. Ask for the total number of people in these six families. How did Student Teachers arrive at that number? What is the average number of people in the six families? (Most likely, they will refer to the traditional algorithm of adding the six numbers and then dividing by six). Ask why this algorithm works.

Then give Student Teachers, working in groups of four, 24 large beans. Have them use crayons, coloured pencils, or markers to colour code their beans:

2 orange, 3 yellow, 3 green, 4 red, 6 purple, 6 blue

Have Student Teachers arrange their beans according to colour, each colour in a vertical line. Then ask them to distribute beans from the larger families (red, purple, and blue) to the smaller families (orange, yellow, and green), trying to make all six lines of beans level.

What do Student Teachers discover? Why did this distribution result in an average of 4?

This 'levelled', 'evened-out', or 'balanced' number is the arithmetic mean. (It is also a number that is part of the dataset. However, the mean could be either a number not in the dataset or not an integer.)

At this point, ask Student Teachers how their levelling the lines of beans connects to the algorithm for finding the arithmetic mean (adding all the entries in the dataset [2 + 3 + 3 + 4 + 6 + 6], and then dividing that sum by 6, the number of entries).

It is likely that Student Teachers have not considered why, in a particular situation, the mode, median, or arithmetic mean might be the preferred measure of central tendency. If so, be explicit in explaining how these measures are dissimilar by using this set of data:

0, 0, 0, 0, 0, 50, 50, 100, 100, 100, 4000

Using ideas from the 'What do Student Teachers need to know?' section, explain how the mean is a useful measure of central tendency when you want an even distribution, but that it is influenced by all the data, including extremes and outliers. In the above dataset, the many 0s and the large outlier have a dramatic effect on the mean (440).

If the above numbers related to earnings, then it would appear that the five persons with 0 income had an income, which is not true. However, if the data represented 11 people sharing their money with each other on the basis of need, each person would receive a 440 share, which would be true.

In situations where the data are skewed, the median may be a more realistic measure because it is not influenced by outliers. Thus, in the above dataset the median is 50—quite a difference from 440.

The mode—the number that occurs most frequently (5 times)—is 0. If the mode were used as the measure of central tendency and this dataset related to earnings, it would show that the unemployed or volunteers were the most typical of the dataset.

Have Student Teachers discuss which measure of central tendency—which 'average'—they think is typical of this dataset. 440? 50? Or 0? Ask why writers might use a particular measure of central tendency to communicate to their audience.

Assignment

To be determined by the Instructor.

Unit 4/week 2, session 3: End-of-course reflection

Activities with Student Teachers

In their end-of-course reflection, Student Teachers will both write about and discuss the course's mathematical content and the way it was presented.

Distribute the 'End-of-Course Reflection' sheet.

Modelling the way the course was designed, Student Teachers will:

- write their individual reflection anonymously
- discuss their thoughts in small group discussion
- engage in a whole-class discussion.

It is especially important to let Student Teachers know that their written reflections and the whole-class discussion will allow you to collect data about both their mathematical learning and their reaction to pedagogy used during the course.

Course Resources

Analysing Children's Thinking in Addition and Subtraction

S

Loretta Heuer

The following questions are designed to assist teacher reflection when analyzing students' thinking about whole-number operations:

1) **Questions about students' thinking: models for addition and different types of addition problems:**

- When joining sets, do they need to begin counting at one, or can they hold the image of a number and 'add on'?
- Can they use Ten Frames to identify patterns of 5, 10, and 1?
- When using a number line, do they count 'hops' rather than points, and relate the process to an addition equation?
- Can they use a balance to model equivalence and then write an equation to represent their work?
- Can they show an understanding of equivalence by generating several addition equations to describe the same number (e.g. $4 + 2 = 6$, $3 + 3= 6$, $5 + 1 = 6$, $6 + 0 = 6$)?

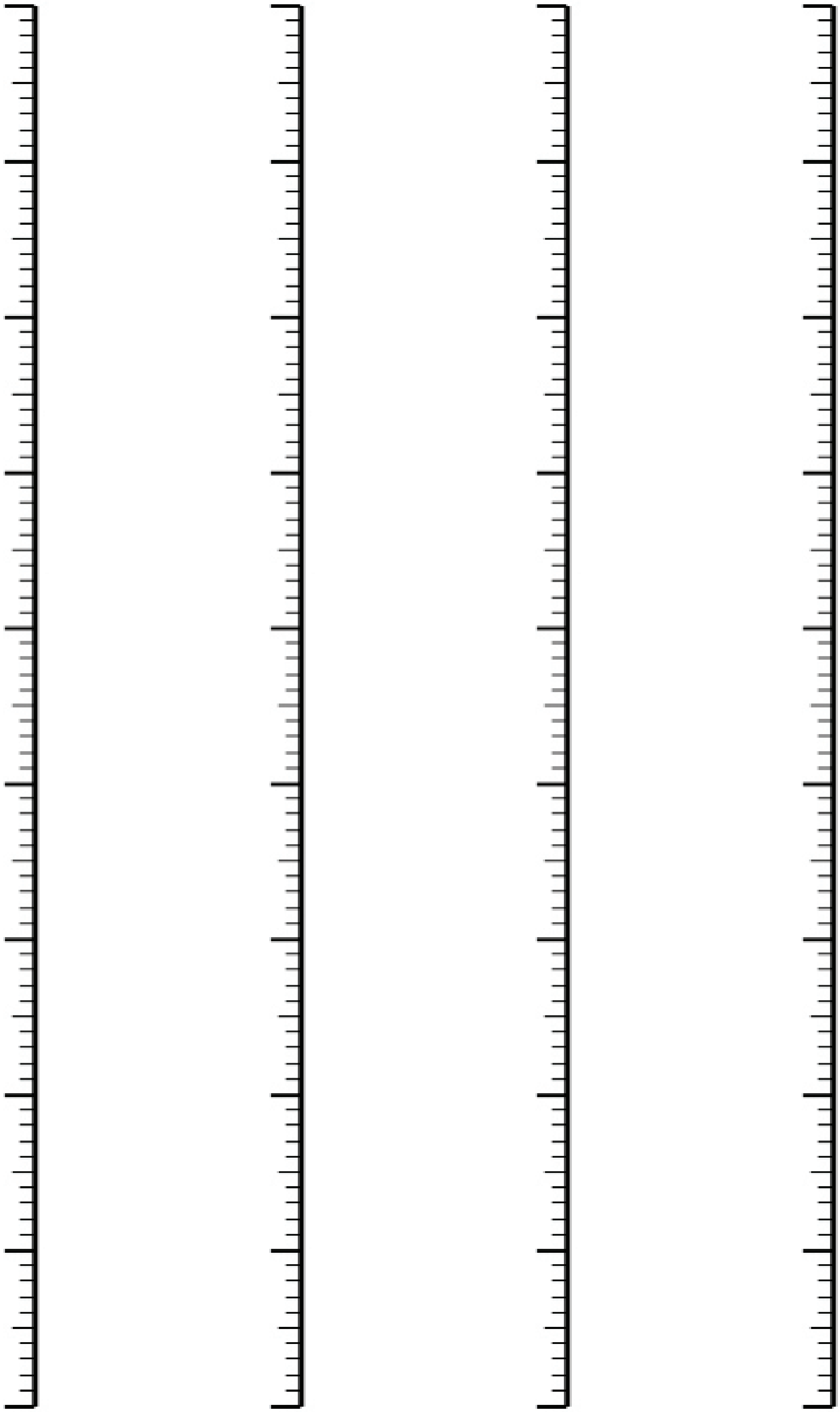
2) **Questions about students' thinking: models for subtraction and different types of subtraction problems:**

- When subtracting, do they begin by counting the whole, or can they hold an image of the whole and 'count back'?
- Can they use models of subtraction other than that of 'taking away'?
- Can they use a number line to model subtraction by 'jumping back' and relating the action to a subtraction equation?
- When comparing two sets, can they describe the difference as more or fewer, and relate that to a subtraction equation?

3) Questions about students' strategic thinking: use of patterns, tools, and derived facts:

- Can students generate 'fact families' that show the relationship between addition and subtraction?
- Do they notice patterns such as doubling, 'near doubling', 'making 10', or adding or subtracting 1 or 0?
- Are they beginning to use the commutative (order) property of addition?
- Can they use a Hundred Chart or number line to show addition by 'counting on'?
- Are they able to create and use an addition chart for the sums of 0 + 0 through 9 + 9?
- Can they see patterns in their addition chart that make remembering their facts easier?
- Which number facts are easy for your students to remember? Why do you think that is so? Which patterns or strategies might help them become more efficient?
- Which number facts are hard for your students to remember? Why do you think that is so? What patterns or strategies might help them become more efficient?

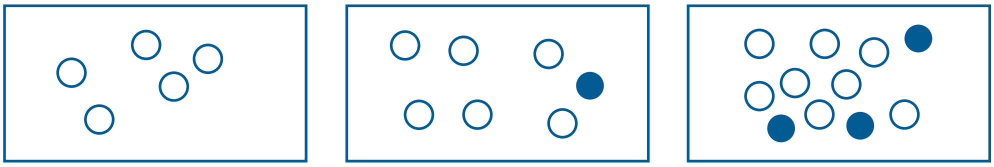
Unit 1, week 3, session 2: Decimal number lines



Subtraction with Integer Chips: Teacher Notes

1) Begin by asking students to make several examples to represent 5 positive chips.

Possible representations:

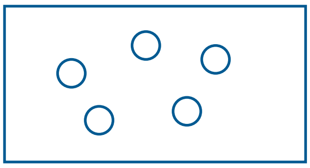


Remind students that a positive-negative pair (a zero-sum pair) is equal to zero.

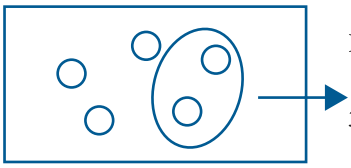
Remind students that adding zero-sum pairs to an amount does not change the amount.

2) Subtraction with the same colour chips:

a) 5 positives take away 2 positives.



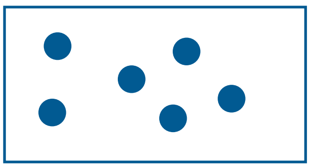
Start with 5 positives.



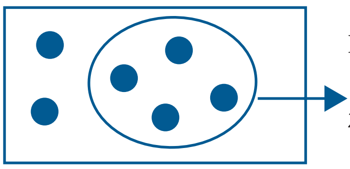
Remove 2 positives.

3 positives remain.

b) 6 negatives take away 4 negatives.



Start with 6 negatives.



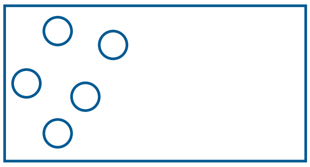
Remove 4 negatives.

2 negatives remain.

These two problems were easy to do because we subtracted a smaller number from a larger number of the same colour.

3) Subtraction with different colour chips (encourage students to look for patterns):

a) 5 positives take away 3 negatives.



Start with 5 positives.

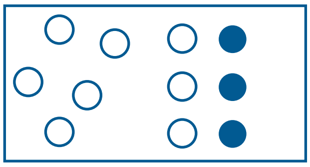
Can I remove 3 negatives? Not yet.

I need to represent 5 positives in a different way.

Add enough zero-sum pairs to remove some negatives.

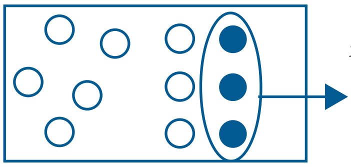
How many zero-sum pairs are needed?

Three, because there are 3 negatives in 3 zero-sum pairs.



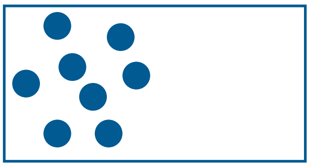
Here is the new representation of the original 5.

Now is it possible to remove 3 negatives?



5 positives minus 3 negatives gives 8 positives.

b) 8 negatives take away 2 positives.



Start with 8 negatives.

Can I remove 2 positives? Not yet.

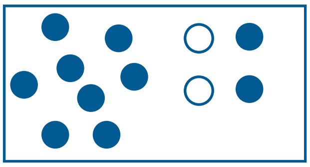
How can I remove 2 positives?

I need to represent 8 negatives in a different way.

Add zero-sum pairs to remove 2 positives.

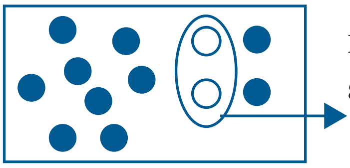
How many zero-sum pairs are needed?

Two, because there are 2 positives in 2 zero-sum pairs.



Here is the new representation of 8 negatives.

Now it is possible to remove 2 positives.



Remove 2 positives.

8 negatives minus 2 positives gives us 10 negatives.

Subtraction with Integer Chips

S

Students should work with integer chips and record actions on this page.

1) Draw three different ways to represent positive 7.

2) Show 7 positive chips and 'take away' 3 positive chips: 7 – 3 = ?

Picture

Describe your picture in words.

Solution

Describe your solution in words.

Translate this problem into an equation using mathematical symbols.

7 positives	minus	3 positives	equals	how many, what sign
__________	______	________	_______	__________________

3) Show 6 negative chips and 'take away' 4 negative chips: -6 – (-4) = ?

Picture

Describe your picture in words:

Solution

Describe your solution in words.

Translate this problem into an equation using mathematical symbols.

6 negatives	minus	4 negatives	equals	how many, what sign
___________	_______	_________	________	__________________

4) Show 7 positive chips and 'take away' 3 negative chips: 7 – (-3) = ?

Picture

Start with 7 positives.

Picture with zero-sum pairs

Add zero-sum pairs. Describe your actions.

Solution

Describe how you arrived at your solution.

Translate this problem into an equation using mathematical symbols.

7 positives	minus	3 negatives	equals	how many, what sign
___________	_______	_________	________	____________________

Compare and contrast the two equations, 7 – 3 = 4 and 7 – (-3) = 10.

5) Show 6 negative chips and 'take away' 4 positive chips: -6 – 4 = ?

Picture		Start with 6 negatives.
Picture with zero-sum pairs		Add zero-sum pairs. Describe your actions.
Solution		Describe how you arrived at your solution.

Translate this problem into an equation using mathematical symbols.

6 negatives	minus	4 positives	equals	how many, what sign
___________	_______	__________	________	____________________

Compare and contrast the two equations, 7 – 3 = 4 and 7 – (-3) = 10.

6) Write a subtraction rule for equations, such as problems 2 and 3, from these notes.

7) Write a subtraction rule for equations, such as problems 4 and 5, from these notes.

You may have noticed in problems 2 through 5 that instead of writing the task as, for example, 'Show 7 positive chips minus 3 positive chips: 7 – 3 = ?', the words 'take away' are used. This phrase is used intentionally, to indicate the 'take away' model for subtraction discussed earlier in the course. This is because when working with integer chips, you literally 'take away' chips from a given set.

End-of-Unit Reflection

Teachers (including Student Teachers) need to reflect on their practice on a routine basis.

This last session of Unit 1, 'Numbers and Operations', is designed to help you reflect on your experience of this unit.

In Unit 1, you worked with many content areas:

- Addition and subtraction of whole numbers
- Multiplication and division of whole numbers
- Fractions, decimals, and per cents
- Operations with fractions and decimals
- Proportion, ratios, and rates
- Integers and operations with integers

At the same time you were also engaging in deep mathematical thinking. As you reflect on each of the four processes below, recall something specific that you found important during this class:

- Modelling and multiple representations
- Mathematical communication
- Problem-solving
- Connections

1) Modelling and multiple representations (a new way to think about a familiar topic)
2) Mathematical communication (you may have heard something in class discussions or observed in children's work samples)
3) Problem-solving (an activity in class in which you were unsure of how to begin—but figured out how to find a solution)
4) Connections (a surprising connection either to real-life situations or to other areas of mathematics: algebra, geometry, or information handling)

Reflection on 'What Do Students Struggle with When First Introduced to Algebra Symbols?'

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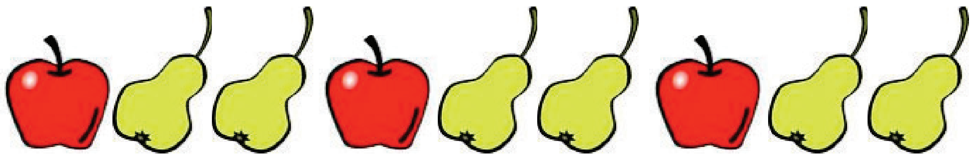
1) How did this article relate to the way you learned algebra as a student? How can x be both 'the unknown' and a variable?
2) What was your first instinct when solving the word problem about sharing money? Did you use arithmetic or algebra? What difference do you see between these two methods? How might you help youngsters begin to shift from arithmetic to algebraic ways of thinking?
3) When you saw the two-column chart and looked for patterns, what did you see? What type of thinking led youngsters to come up with other patterns? Were those patterns valid? Why or why not?
4) When you saw the growth pattern in Figure 2, how did you extend the pattern to find the number of sticks in Figure 25?
5) What do you think of the author's comment that 'moving from arithmetic to algebraic generalizations is a process that has been found to take time'? If this is so, when should the 'algebraicifacation' of arithmetic begin?
6) How does the difference between the minus sign for the operation of subtraction and the negative sign for numbers less than 0 influence students' work in algebra?

Patterns in Numbers and Shapes

Patterns are so much a part of our daily lives that we rarely stop to consider them. By having some sense of regularity and predictability, our lives are made more manageable. Similarly, patterns, which are the foundation of mathematics, can make learning mathematics more manageable for our students.

The algebraic thinking of young children is rooted in the repeating patterns of songs, stories, and physical activities.

In grade 1, children may have needed to be introduced to simple repeating visual patterns in a series of pictures, such as:

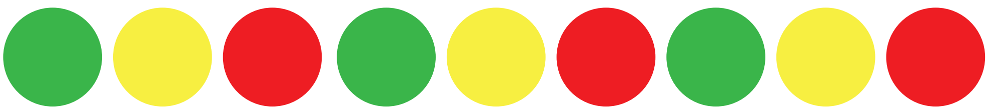


When children were duplicating linear patterns in grade 1, they were preparing themselves for later work: extending a given linear pattern. To extend or continue a pattern, children need to identify and isolate the 'pattern unit', or repeating element. In the above linear pattern, the repeating unit would be red-green-green or apple-pear-pear. Once this has become clear to children, they will begin to understand how adding another pattern unit, and another, and another can extend a linear pattern indefinitely.

After having copied and extended linear patterns, children will explore patterns in two dimensions (such as in fabric) and growth patterns.

The simplest growth pattern is our basic counting pattern: + 1. When counting, we add one to a number to get the next number. When we 'count by twos' starting at 0, the + 2 rule creates the sequence of even numbers: 0, 2, 4, 6, 8. … However, if we skip-counted by twos starting at the number 1, the sequence would be the odd numbers: 1, 3, 5, 7, 9. …

When young children identify and isolate the pattern *unit*, they will generalize a pattern *rule* that they can translate into other formats. In the following pattern, the green-yellow-red sequence of the traffic light would be repeated again and again if we stood on a street corner. Our eyes would see the following sequence of coloured lights.



However, if we wanted to describe what we saw, we could translate those images into words:

green-yellow-red green-yellow-red green-yellow-red

If we wanted to further simplify the pattern, we could encode the sequence by using the initial letter of each colour:

G-Y-R G-Y-R G-Y-R

Then, if we wanted to generalize the pattern to *any* colour or image, we could say that the repeating unit was:

ABC ABC ABC

We can use this repeating ABC-ABC-ABC rule to create a pattern using new images, such as rose-tulip-daffodil:

This same ABC pattern unit could be translated into physical actions by asking children to touch their head-waist-toes, head-waist-toes, head-waist-toes. …

Although part of the 'Algebra' unit, pattern work needs to be connected to work that Student Teachers performed in the 'Numbers and Operations' unit. Make sure to point out patterns in addition and subtraction that were alluded to in the 'Numbers and Operations' unit. Note that when exploring place value, children focus on patterns of tens and ones. Take every opportunity to highlight patterns in multi-digit numbers.

Why is this important in the early grades?

Young children's work with patterns is their work with algebraic thinking. Young as they are, children are creating the algebraic ground floor that will support all their future work with patterns and functions.

However, algebraic thinking is not simply a long-term issue. In the short term, young children will utilize patterns in their number charts and on their number lines to make sense of addition, subtraction, and work with multi-digit numbers. Without relying on patterns, learning number facts and working with place value is a daunting task! The role of patterns, as mentioned earlier, is to make life and mathematics more manageable.

Attention to patterns will help children begin thinking about generalizations. Is there a pattern here? Can it be continued? Can it be translated? Will it work in other circumstances? Questions such as these move children from the specific to the general, an important conceptual and developmental step in mathematical thinking.

Finally, when solving a problem, the first thing we tend to do as adults is to look for a pattern—primarily because it tends to be a time-efficient (and time-tested) strategy. Similarly, children need to know that patterns are more than attractive visual arrangements or a useful device when computing sums. They need to learn that a 'look for patterns' mindset is an important strategic tool when solving problems in all strands of mathematics.

The Coin Graph

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Preparation:

- Collect an empty jar and two coins for each student in class.
- On large chart graph paper, prepare a T-chart with columns labelled 'Number of People' and 'Number of Coins'.
- On another piece of chart paper, draw the two axes for a first quadrant graph and label them as 'Number of People' and 'Number of Coins'.

When students arrive in class, give each student two coins.

Begin by passing the jar to the first student and have him or her put the two coins in the empty jar.

Record this on the T-chart as 1 person, 2 coins.

Continue passing the jar around the class, with students putting coins in and you recording the number of coins in the jar.

When this data-collecting activity is finished, turn students' attention to the graph. How could the data be entered? If necessary, review how to scale the axes and how to plot points, with each point resulting from the ordered pair on the T-chart (people, coins).

After plotting the points, ask what the students notice. Is there a pattern? Could they extend the pattern to reflect someone arriving in class late who needed to put two more coins in the jar? What might be a pattern rule for what is shown in the table and on the graph?

Finally, ask if the points should be connected. If there is general consensus that the points should be connected, ask if there could be a point plotted halfway between the points (2,4) and (3,6).

(Yes, there could be 5 coins, but that would imply 2.5 people. Is that possible?)

Explain that most graphs that result from counting objects are called *discrete graphs* and the points should not be connected.

Graph Stories

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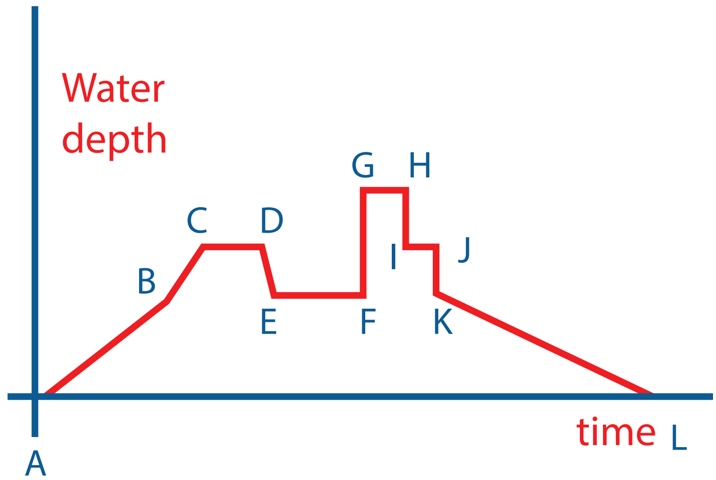
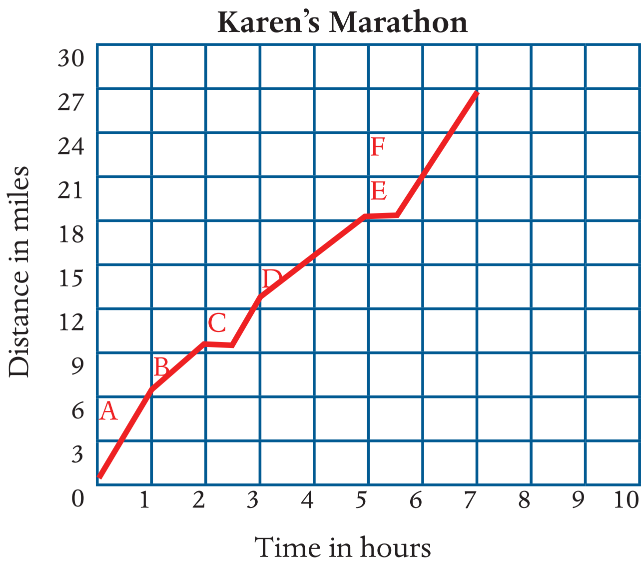
Graph A



1) What does this graph imply about water depth over time for any interval and time between intervals?
2) Should the points on the graph have been connected?
3) What real-life situation might be modelled by this graph?

Graph B



1) What is the rate of change for interval 'A'?
2) Explain what you think may have happened during interval 'C'.
3) How long is a marathon?

Kitchry (Moong Dal Rice) Recipe

For 4 servings:

200 grams basmati rice
100 grams yellow split moong dal
1 tablespoon of ghee
1 teaspoon black mustard seeds
1 teaspoon cumin seeds
1 small piece of ginger, minced
½ teaspoon turmeric (curcuma)
½ teaspoon salt
900 millilitres water
black pepper
fresh coriander

Wash the rice and moong dal well. Put the ghee in a saucepan and heat, then add the mustard seeds, cumin seeds, and ginger. Stir for a moment until you see the seeds pop. Add the rice, moong dal, turmeric, and salt and stir well until blended with all the spices. Add the water and bring to a boil. Boil for about 5 minutes, uncovered, stirring everything occasionally. Then turn down the heat to low and cover. Cook until ready for 20 to 25 minutes. Season with freshly ground black pepper, and chop some coriander and sprinkle over the finished dish.

Think about the amount of ingredients in this recipe designed for four servings.

What if you wanted to serve only one or two people? How would that change the amount of each ingredient?

On the other hand, if you have a large family you might want to make 8 or even 16 servings. How would that change the amount of ingredients?

Create a T-chart of how much moong dal you would need to prepare this recipe for 1, 2, 4, 8, and 16 people.

Create another T-chart chart that shows how many cardamoms you would need to prepare the recipe for the same number of people.

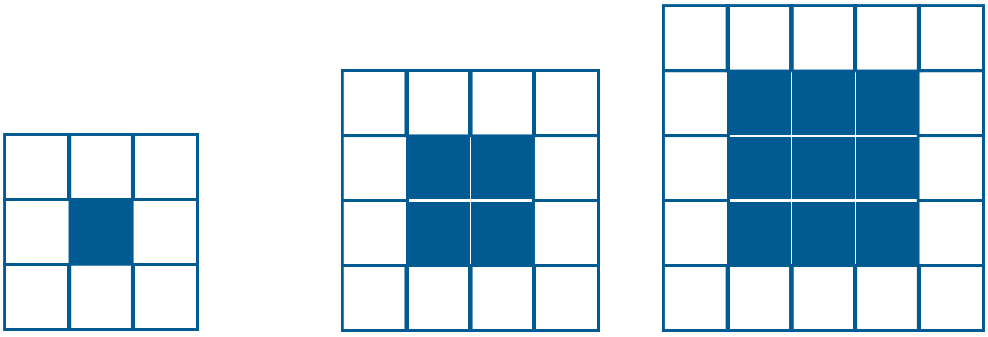
What do you think is a pattern rule for each?

Translate your two charts into graphs.

How did you scale your axes? What patterns do you see? Is one graph continuous and the other discrete? How do the phrases 'how much' and 'how many' differ mathematically? What do they mean when graphing a pattern rule?

Tiling the Pool

You are designing square swimming pools. Each pool has a square centre, which is the area of the water. You are using tiles to surround the pools. The first three pools are shown below.



Pool 1 Pool 2 Pool 3

Sketch several more pools on graph paper.

Determine the number of tiles you need to surround each pool, and then look for patterns in the table.

Pool number	Number of water tiles	Number of border tiles	Total number of tiles
1			
2			
3			
4			
5			
6			
n			

Create a graph that shows how the number of border tiles change as the pool size changes. (Is this graph discrete or continuous?)

Find a symbolic expression that can be used as a generalized rule to describe the pattern. Try to find at least two more.

Consider the following questions:

1) If there are 36 water tiles, how many border tiles are there?
2) Tell me about the 50th pool.
3) What generalizations did you determine?
4) Which representations did you use during this activity (language, pictures, numeric, or symbolic)?
5) What questions might a teacher ask students to prompt their thinking?

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Taxi Fares

Welcome to Dubai!

This is your first visit to Dubai, and so you think you should travel by taxi.

Before you go, you look up taxi fares on the Internet and find the following costs:

- 6 Dirhams upon entering the cab
- 1.50 Dirhams for each kilometre travelled

Use what you know about algebra to create a table, graph, and symbolic expression to show Dubai taxi fares from your hotel to the following destinations:

- The beach: 2 kilometres from your hotel
- Gold Souk: 1 kilometre from your hotel
- Your friend's house: 3 kilometres from your hotel
- The airport: 4 kilometres from your hotel

Attributes of Polygons

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Which of the following 25 polygons fit these definitions or have the following characteristics?

Group 1

1) A pentagon
2) Opposite sides parallel
3) At least one right angle

Group 2

1) A hexagon
2) All sides congruent
3) Regular polygon

Group 3

1) An octagon
2) Opposite angles congruent
3) Parallelogram

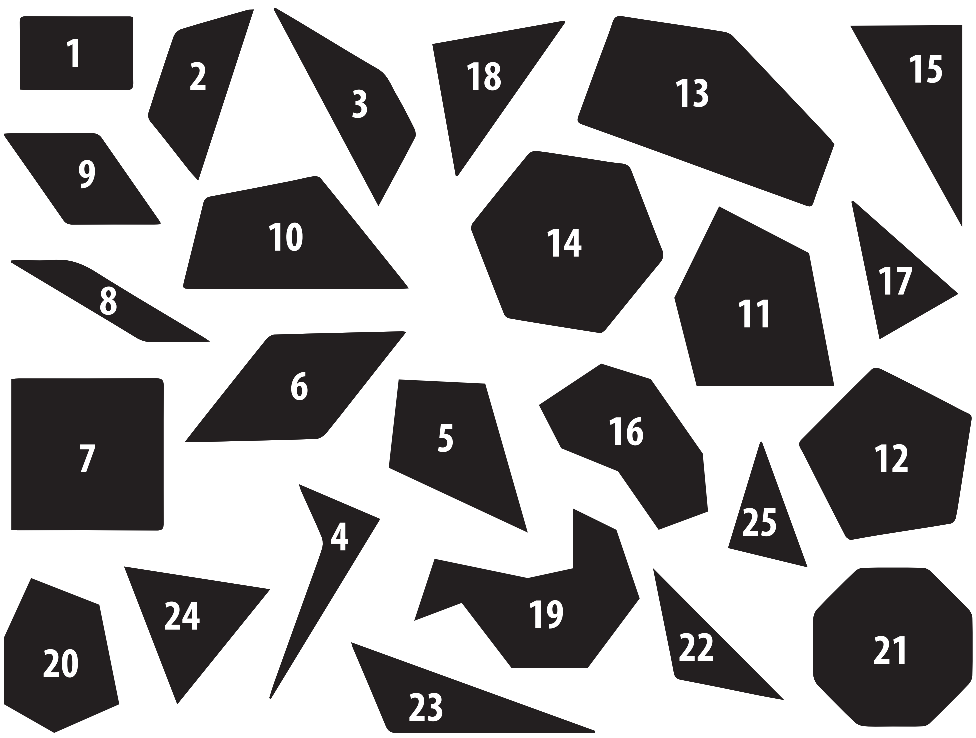
Group 4

1) A rhombus
2) A quadrilateral
3) Opposite sides congruent

Group 5

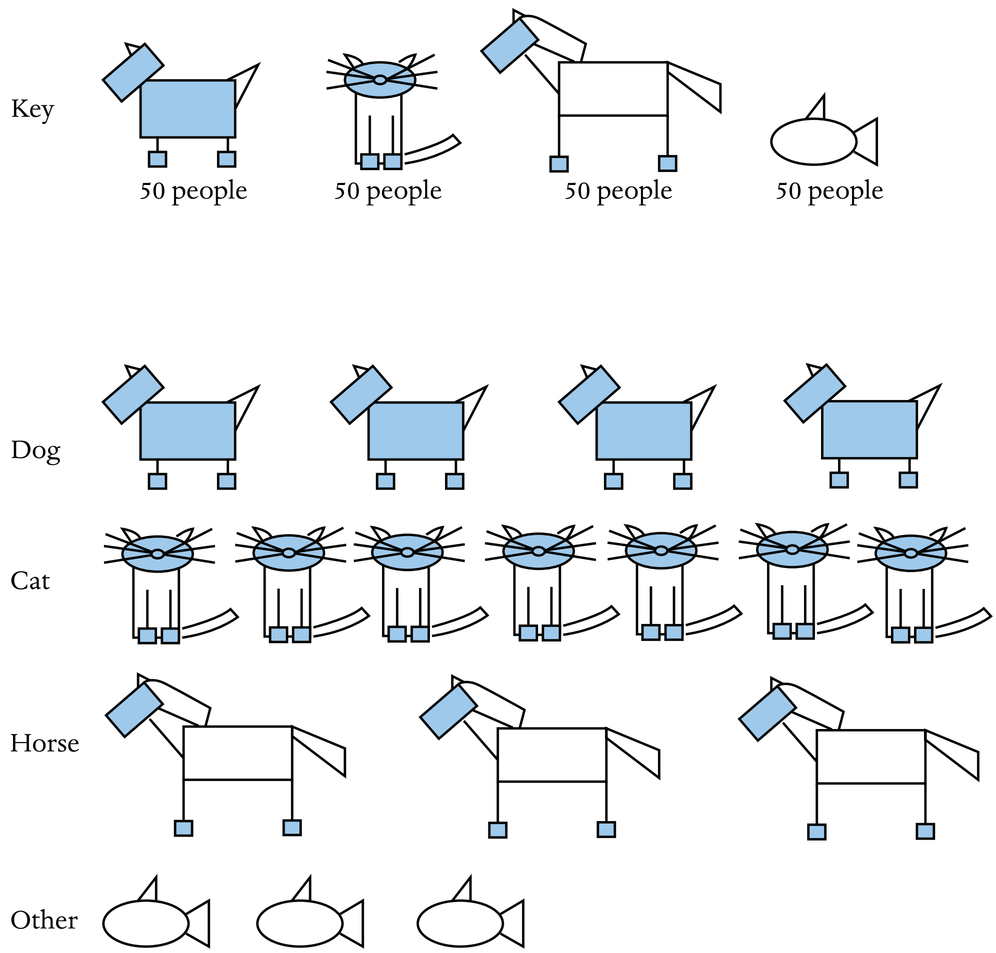
1) A trapezoid
2) All angles congruent
3) A concave polygon

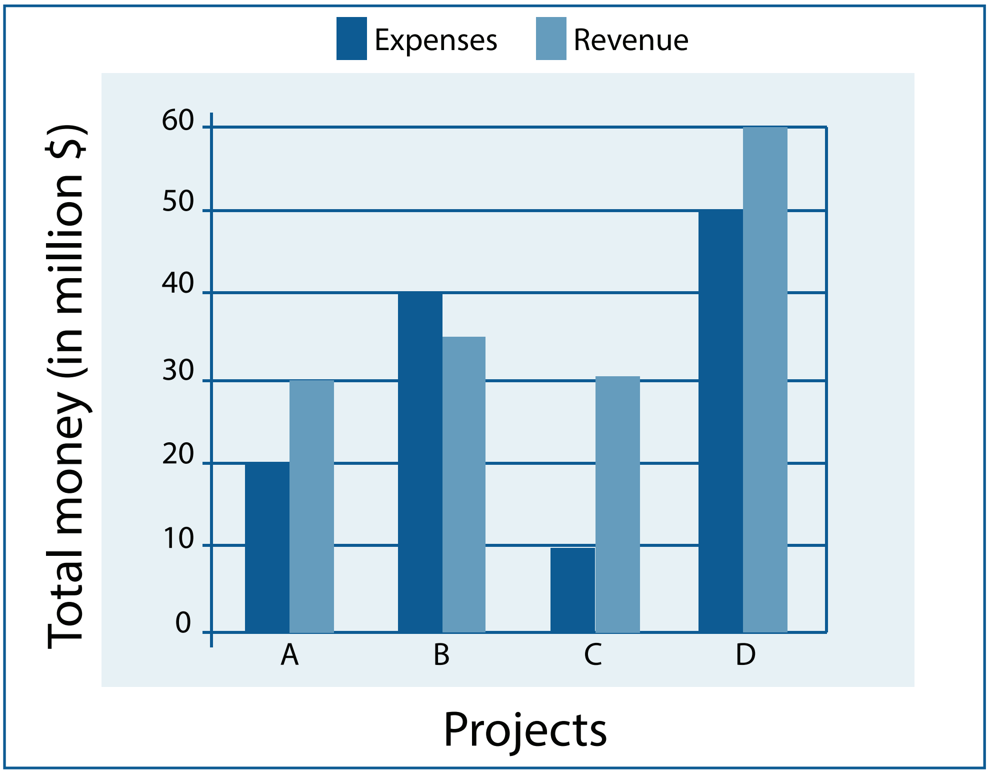
25 polygons



Graph Analysis 1

How clear are this pictograph, double bar graphs, and line plot in communicating information? What questions seem to be addressed by each of these graphs?





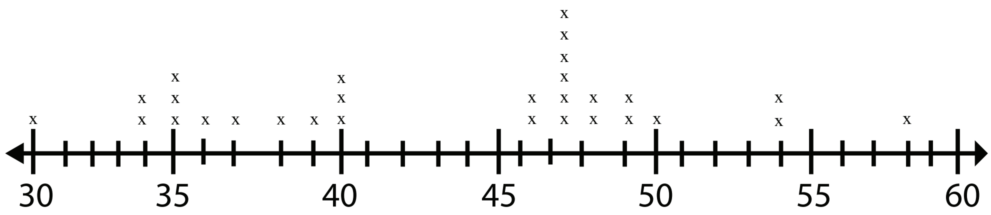


Figure: Line plot

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Graph Analysis 2

First, analyse these two graphs with a partner.

Then, before reading the questions that follow the graphs that others have suggested, pose at least three questions of your own that you think each graph could answer.

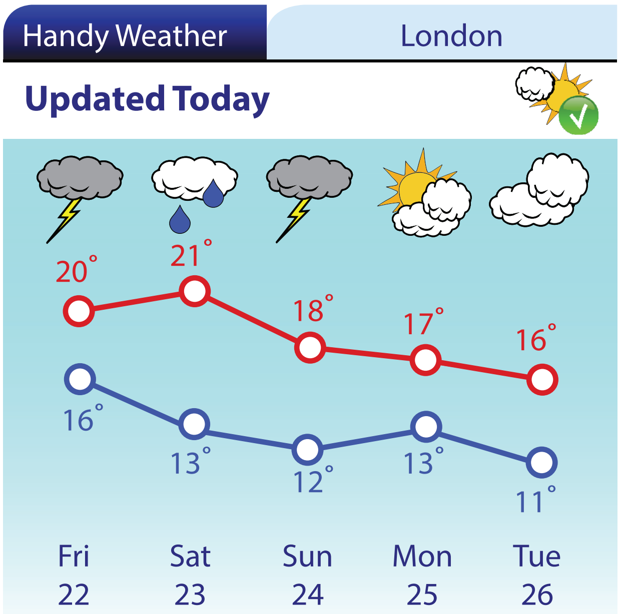


Figure: Line graph

Questions about the weather map:

1) Do you think there might have been a small change in temperature between the data points on this graph?
2) Is the line added to the graph for the viewer's convenience?
3) How is a line graph designed for general communication different from a linear function graph?
4) Do weather services use different intervals on various pages of their website? Which interval would you need to decide what to wear in a few hours? Which interval would you need to know what the temperature will be three days from now?
5) What do you predict the temperature will be on Wednesday?
6) What do the red and blue colours of the lines mean?
7) Are these colours unique to this site or do they have a more universal connotation?

Here is a scatter plot that shows the size of a plant in relation to its age.

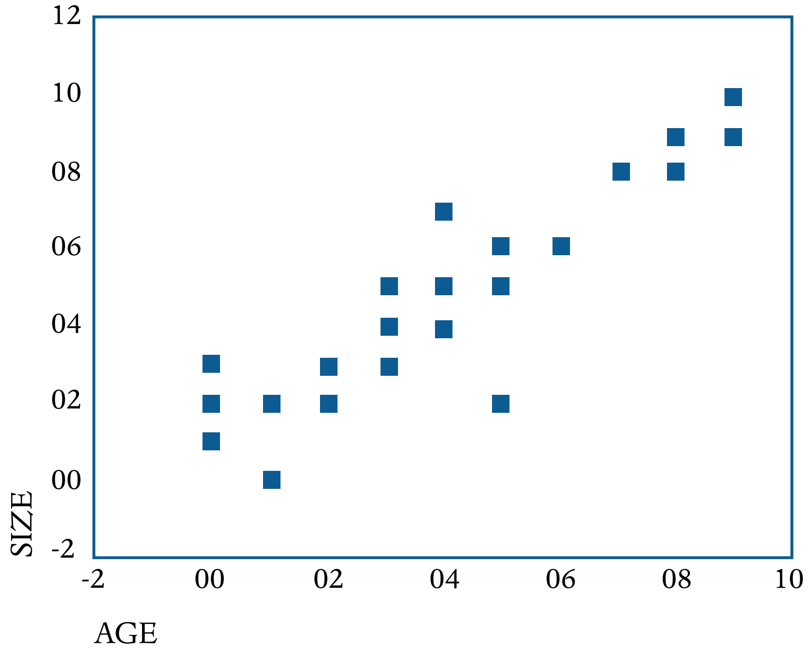


Figure: Scatter plot

Questions about the scatter plot of plant growth:

1) Why does the graph start below the (0,0) point?
2) What about the three plants at age 0?
3) Do you notice a trend?
4) How many plants were involved in this study?
5) What sort of correlation do you notice?
6) Does it suggest a trend line?
7) Without doing formal statistical calculations, where would you place a 'line of best fit'?

Frequency Tables and Mode

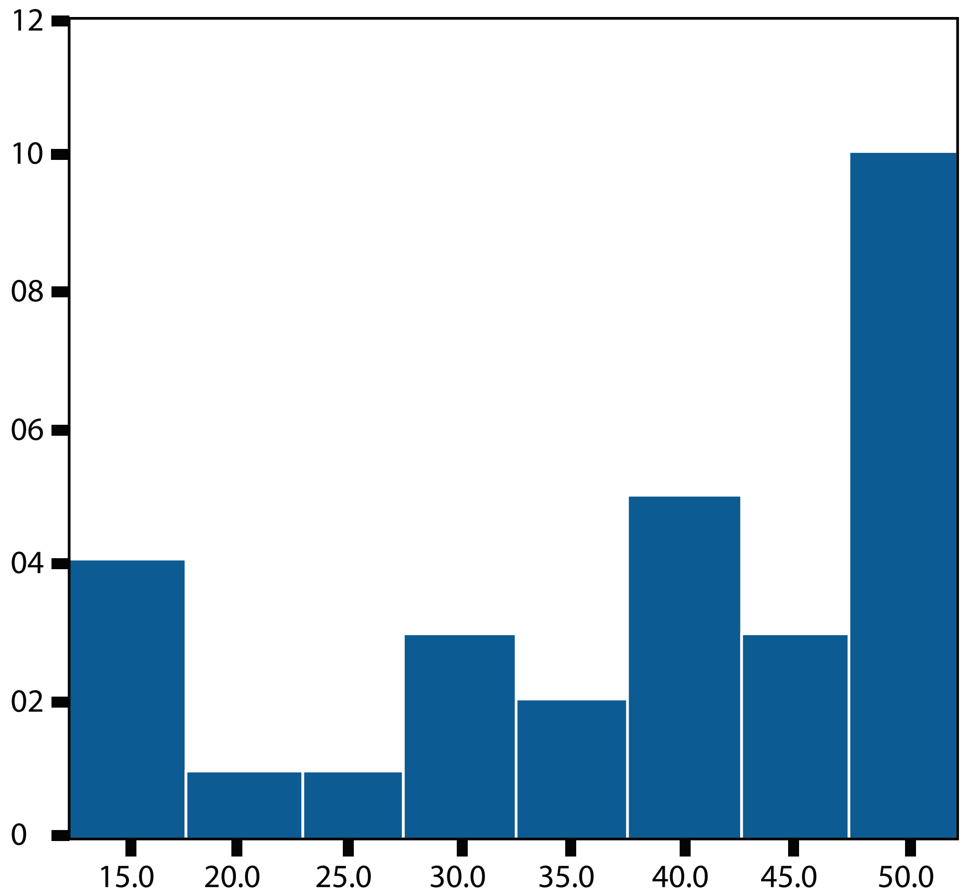
1) Consider this dataset. How will you turn it into a frequency table?

4, 5, 8, 5, 7, 8, 9, 8, 8, 7

Create your frequency table now.

2) Consider this graph. Which bars occur most frequently? Which number occurs most frequently?

What is the mode for this dataset?



Frequency Tables and Mean

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1) Create a frequency table in ascending order for the following dataset:

40, 51, 41, 23, 33, 40, 58, 73, 32, 32, 41, 52, 77, 25, 50, 67, 41

Why is organizing the data into ascending order important?

What patterns do you notice?

Where does the data 'cluster' into 'intervals'?

Explore measures of central tendency and distribution. What is the:

- 'Modal interval'
- Median
- Mode
- Range (expressed in words and as a number)
- Outliers

How did you find each of these?

How could you find the mean of this dataset?

2) The following dataset shows the number of family members in several households:

2, 3, 3, 4, 6, 6

Use colour-coded manipulatives and/or drawings to show how you can decompose some of the greater numbers and redistribute them to some of the lesser numbers to equalize all six numbers in the dataset.

2 orange, 3 yellow, 3 green, 4 red, 6 purple, 6 blue

How does this hands-on and visual model relate to the traditional algorithm for finding the arithmetic mean?

End-of-Course Reflection

During this course you worked through the four units below.

1) Think back. What stood out for you in each of the four units? What did you find new or interesting? Did you re-think the way you had thought about a particular concept?

- Numbers and operations:

- Algebra:

- Geometry and geometric measurement:

- Information handling:

2) Did you learn maths better as a result of taking this course? If so, why do you think this happened? Which mathematical topics do you feel you understand more deeply now?

3) Are you more comfortable with mathematics (and the idea of teaching mathematics to children) after having taken this course? Why or why not?

4) Reflect on the way the class was structured (whole-class introduction to the topic, group work, manipulative materials, multiple representations, small group and whole-class discussions, etc.).

   Was this course different from other maths courses you have taken? If so, in what ways?

   Did the course's interactive lesson design help you have a deeper mathematical understanding of certain topics? Which ones?

5) When you become a teacher, which ideas from this course will you bring into your classroom?

Appendix

PROFESSIONAL STANDARDS FOR TEACHING MATHEMATICS

Professional Standards for Teaching Mathematics

In 2009, the Ministry of Education passed into policy a set of National Professional Standards for Teachers in Pakistan (NPSTP). The 10 standards describe what a teacher should know and be able to do.

The following is a list of standards specific to the teaching of mathematics. They were developed to be used in conjunction with the three mathematics courses in the B.Ed. (Hons) Elementary/ADE programme. They provide a description of the knowledge, skills, and dispositions a teacher requires to teach mathematics.

This set of standards for teaching maths is linked to the NPSTP. The first standard in the NPSTP concerns Subject Matter Knowledge: the knowledge, skills, and dispositions a teacher requires to teach the National Curriculum. In the NPSTP, knowledge, skills, and dispositions are described in general terms for all subjects. Here, they are described specifically for teaching mathematics.

The standards for teaching mathematics may be used by Instructors and Student Teachers in a variety of ways, such as for assessment (including self-assessment) and planning instruction. The standards may also be used as part of instruction. Helping Student Teachers deconstruct and understand the standards (and how they apply in the classroom) will help them learn about teaching mathematics.

Subject Matter Knowledge (Teaching Mathematics)

Knowledge and understanding

Teachers know and understand the following:

- the National Curriculum framework for mathematics
- the basic concepts and theories of maths; the history and nature of mathematics and how to explore mathematical relationships, to represent data visually and symbolically, and to make generalizations that predict future outcomes; and the structure and process of acquiring additional knowledge and skills in mathematics
- in depth knowledge of mathematics and its relationship to other content areas
- the relationship between mathematics and other disciplines
- the usability of mathematics in everyday life
- that mathematics is not a discrete set of facts, but a relationship between a wide range of concepts across the five mathematical strands
- that maths content and pedagogy cannot be separated and that an effective lesson plan makes use of content and pedagogy simultaneously
- the evolving nature of mathematics and its content
- that learning mathematics is an ongoing and reflective practice.

Dispositions

Teachers value and are committed to doing the following:

- facilitating the construction of knowledge in multiple ways
- deepening learners' understanding and skills
- making mathematics integral for application to real-world situations
- recognizing and building on the diverse talents of all students and helping them develop self-confidence and mathematical literacy
- believing that all students can make meaning in mathematics through their own constructions and be successful users of mathematical knowledge and skills
- acknowledging the intellectual learning and growth of all students (boys and girls)
- creating an engaging learning environment that promotes knowledge building through student interaction (discussion, asking questions, and sharing ideas and examples) with peers and teachers
- providing meaningful learning experiences that deepen maths content knowledge (rather than emphasizing rote memorization of isolated facts)
- teaching students to think critically about information and how to make informed decisions that affect their lives.

Performance and skills

Teachers demonstrate their knowledge and understanding by doing the following:

- selecting mathematical content from the National Curriculum that reflects and extends students' prior mathematical knowledge and understanding
- using tools and methodologies for inquiry appropriate to mathematics and the content being taught
- effectively explaining mathematical content from multiple perspectives and in multiple ways that are appropriate for the students they are teaching
- giving examples of how mathematical content applies to daily life
- asking engaging questions that stimulate thinking and reasoning rather than the memorization of facts
- using of maths manipulatives to deepen students' understanding of mathematical concepts
- modelling ways to help students construct their knowledge of mathematics
- analysing materials (e.g. textbooks) to see if they are useful and appropriate resources for teaching and learning mathematical concepts and skills detailed in the National Curriculum
- arousing curiosity about mathematical ideas
- organizing opportunities for students to discuss and develop their understanding of mathematical concepts.

Adapted from National Professional Standards for Teachers in Pakistan. Policy and Planning Wing, Ministry of Education, Government of Pakistan (2009). Available from:

➢ http://unesco.org.pk/education/teachereducation/files/National%20Professional%20Standards%20for%20Teachers.pdf

References

Australian Association of Mathematics Teachers, *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006).

Douglas Grouws, ed., *Handbook of Research on Mathematics Teaching and Learning* (Reston, VA: National Council of Teachers of Mathematics, 2003), 111.

Jan de Lange, *Mathematical Literacy for Living OECD-PISA Perspective* (Utrecht: Freudenthal Institute, Utrecht University, 2006).

Mathematical Association of America, *The Mathematical Education of Teachers* (Washington DC: Conference Board of Mathematical Sciences, 2001).

M. Stone Wiske, *Teaching for Understanding: Linking Research with Practice* (San Francisco: Jossey-Bass Publishers, 1998).

Higher Education Commission